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Let G be a non-empty set and 'o 'denotes a binary operation defined on G, then the algebraic structure $\langle G,o\rangle$ is called a groupoid.

Examples: $\langle N,+\rangle$, $\langle Z,+\rangle$, $\langle Q,+\rangle$, $\langle R,+\rangle$. $\langle C,+\rangle$, $\langle N,\cdot\rangle$, $\langle Z,\cdot\rangle$, $\langle Q,\cdot\rangle$, $\langle R,\cdot\rangle$ and $\langle C,\cdot\rangle$ are groupoids . The symbols + and '$\cdot$ 'denote ordinary addition and multiplication respectively.

$\langle N,-\rangle$ is not groupoid because substraction of two natural numbers is not necessarily a natural number i.e. $2\in N$ and $6\in N$ but $2-6=-4\notin N$.

$\langle Z,-\rangle$. $\langle Q,-\rangle$. $\langle R,-\rangle$ and $\langle C,-\rangle$ are groupoids. But $\langle Z,\div\rangle$, $\langle Q,\div\rangle$, $\langle R,\div\rangle$ $\langle C,\div\rangle$ and $\langle N,\div\rangle$ are not groupoids as division is not binary operation on any these sets.

Semi-Group:

A groupoid $\langle G,o\rangle$ is called a semi- group if the binary operation satisfies associative law in G, i.e.

(aob)oc = ao(boc) for every $a,b,c\in G$

A semi group (G. o) is called commutative semigroup if the operation o is commutative in G, i.e.

aob = boa for every $a,b\in G$

Example : $\langle N,+\rangle$, $\langle Z,+\rangle$, $\langle Q,+\rangle$, $\langle R,+\rangle$. $\langle C,+\rangle$, $\langle N,\cdot\rangle$, $\langle Z,\cdot\rangle$, $\langle Q,\cdot\rangle$, $\langle R,\cdot\rangle$ and $\langle C,\cdot\rangle$ are semi -Groups, these are also commutative semi group. The operation substraction and division do not satisfy associative law in any set of number.

Monoid:-

A groupoid $\langle G,o\rangle$ is called Monoid or semi- group with identity if

(aob)oc = ao(boc) for every $a,b,c\in G$

and there exists an identity say $e\in G$ such that

aoe = eoa for every $a\in G$

then e will be called an identity of G with respect to 'o'.

A monoid is called "commutative monoid" or "commutative semigroup with identity", if operation is commutative.

Group :

Let $\langle G, o \rangle$ be a groupoid then it is called a group if it satisfies following properties;

1. **Assosiative Law :**

   (aob)oc = ao(boc) for every $a, b, c \in G$

2. **Existence of Identity:** There exists an element e €G such that

   aoe = eoa for every $a \in G$

3. **Existence of Inverse :** For each element $a \in G$, there exists an element $b \in G$ such that

   aob = boa = e

the element b is called the inverse of a with respect to binary operation o.

A group $\langle G, o \rangle$ is called "Commutative group or Abelian group" if the binary operation ' o ' is commutative in G.

If a group $\langle G, o \rangle$ contains finite number of elements then the group is called a "Finite group" otherwise it is called an "Infinite group". If G contains n distinct elements, then G is called "Finite group of order n".

Question: Prove that cube roots of unity forms an abelian group with respect to multiplication.

Solution: Let $G = \{1, \omega, \omega^2\}$, the set of cube roots of unity. We form a composition table which is also known as "Caley Composition table" for given algebraic system --

.	1	ω	ω^2
1	1.1 = 1	1. $\omega = \omega$	1. $\omega^2 = \omega^2$
ω	ω .1 = ω	$\omega . \omega = \omega^2$	$\omega . \omega^2 = \omega^3 = 1$
ω^2	$\omega^2 .1 = \omega^2$	$\omega^2 . \omega = \omega^3 = 1$	$\omega^2 . \omega^2 = \omega^4 = \omega$

Closure Law:

All the entries of composition table are elements of G i.e.

$1 \,.\, \omega = \omega \in G$

$\omega \,.\, \omega^2 = \omega^3 \in G$

$\omega^2 \,.\, 1 = \omega^2 \in G$ etc.

Associative Law: The elements of G are complex numbers and we know that the multiplication of complex numbers is associative.

Existance of Identity: $1 \in G$ is the identity element.

Existance of Inverse : $1, \omega^2, \omega$ are the inverse of $1, \omega, \omega^2$ respectively i.e. each element of g has its inverse in G .

Therefore $\langle G, o \rangle$ is a group. Since elements of G are complex numbers and we know that the multiplication of complex numbers satisfies commutative law, so $\langle G, o \rangle$ is an abelian group.

Similarly, we can prove that fourth and n^{th}-roots of unity forms an abelian group with respect to operation of multiplication.

Properties of group:

(i) Identity element of a Group is unique.

(ii) Every element of a group $\langle G, o \rangle$ has unique inverse in G.

(iii) In a group $\langle G, o \rangle$ $(aob)^{-1} = b^{-1} o a^{-1}$ $\forall \; a, b \in G$

(iv) A group $\langle G, o \rangle$ is commutative iff $(aob)^{-1} = a^{-1} o b^{-1}$ $\forall \; a, b \in G$

Left and Right cancellation Law: In a group $\langle G, o \rangle$

Left Cancellation Law $aob = aoc \quad \Rightarrow \quad b = c \qquad \forall \; a, b, c \in G$

Right Cancellation Law $boa = coa \quad \Rightarrow \quad b = c \qquad \forall \; a, b, c \in G$

Subgroup:

Theorem:- A non -empty subset H of a group $\langle G, o \rangle$ is a subgroup iff

(i) $a \in G, \; b \in G \quad \Rightarrow \quad aob \in H$

(ii) $a \in H \quad \Rightarrow \quad a^{-1} \in H$

Theorem:- A non -empty subset H of a group $\langle G, o \rangle$ is a subgroup iff

$$a \in H, \; b \in H \quad \Rightarrow \quad aob^{-1} \in H$$

Theorem: Intersection of two subgroups of a group is also a subgroup.

Proof: Let H_1 and H_2 be any two subgroups of G. Then $H_1 \cap H_2 \neq \phi$, since at least identity e is common to H_1 and H_2.for H_1 and H_2 is a subgroup it is sufficient to prove that

$$a \in H_1 \cap H_2, \quad b \in H_1 \cap H_2 \quad \Rightarrow \quad ab^{-1} \in H_1 \cap H_2$$

Now $\quad a \in H_1 \cap H_2 \quad \Rightarrow \quad a \in H_1 \quad$ and $\quad a \in H_2$

$b \in H_1 \cap H_2 \quad \Rightarrow \quad b \in H_1 \quad$ and

But H_1 and H_2 are two subgroups, therefore

$$a \in H_1 \quad b \in H_1 \quad \Rightarrow \quad ab^{-1} \in H_1$$

$$a \in H_2 \quad b \in H_2 \quad \Rightarrow \quad ab^{-1} \in H_2$$

i.e. $\quad ab^{-1} \in H_1$ and $ab^{-1} \in H_2 \quad \Rightarrow \quad ab^{-1} \in H_1 \cap H_2$

therefore $H_1 \cap H_2$ is subgroup of G.

But union of two subgroups is a subgroup iff one is contained in the other.

Complex of a group:-

Any non-empty subset H of G is called a complex of the group G.

If H and K are two complexes of group G,then

$$HK = \{x \in G, x = hk, h \in H and k \in K\}$$

i.e. HK is a complex of G consisting of the elements of g on multiplying each member of H with each member of K. Multiplication of complexes is associative.

Inverse of a complex:

Let H be any complex of G, then

$$H^{-1} = \{h^{-1} : h \in H\}$$

i.e. H^{-1} is the complex of G consisting the inverse of element of H.

Questions:

(1) The set G of all complex numbers of modulus unity forms a multiplicative abelian group.

(2) Prove that fourth roots of unity forms an abelian group with respect to multiplication.

(3) Prove that n^{th} roots of unity forms an abelian group with respect to multiplication.

(4) If Q^+ is the set of all positive rational numbers and 'o' an operation on Q^+ defined as $aob = \frac{a}{b}$ $\forall\ a,b \in Q^+$, then (Q^+, o) is not a group.

(5) If H and k are any two complexes of group G then $(HK)^{-1} = K^{-1}H^{-1}$.

Reference :

1. Modern Algebra by A.R. Vashistha
2. Abstract Algebra by Khanna & Bhambri
3. Algebra & trigonometry by Pandey

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