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1. Unit**: Four**
2. Topic**: Algebra**
3. Sub topic**: Normal Subgroup**
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Homomorphism:

Let G and $\overline{G}$ be two groups, then a map from G into $\overline{G}$ is called group homomorphism iff

$$f(ab) = f(a).f(b) \qquad \forall a,b \in G$$

- A homomorphism f from G into $\overline{G}$ is called an isomorphism if f is one-to-one and represented by $G \approx \overline{G}$.
- A homomorphism of a group G into G itself is called an Endomorphism of G.
- An isomorphism of a group G onto itself is called an automorphism of G.

Theorem: The relation $\approx$ of being isomorphic to in the set of all groups is an equivalence relation.

Proof: Let G, G'. G'' ………be groups.

Reflexive: The identity map $I: G \to G$ is an isomorphism, we have $G \approx G$ for every group G.

$\Rightarrow$ the relation $\approx$ is reflexive.

Symmetry: Let $G \approx G'$, then there exists a map $f: G \to G'$ which is an onto isomorphism.

Thus $f^{-1}: G' \to G$ is bijection, since f is bijection.

Now f^{-1} is a homomorphism, For if $f^{-1}(a') = a$ and $f^{-1}(b') = b$

where $a', b' \in G'$ and $a, b \in G$

then $f(a) = a'$ and $f(b) = b'$

Therefore $f^{-1}(a'b') = f^{-1}\{f(a)f(b)\}$

$= f^{-1}\{f(ab)\}$ since f is a homomorphism

$= ab$

$= f^{-1}(a').f^{-1}(b')$

Thus f^{-1} is an isomorphism which is also onto

Hence $G \approx G' \Rightarrow G' \approx G$

i.e. the relation $\approx$ is symmetric.

Transitive: Let $G \approx G'$ and $G' \approx G''$ then there exists map $\phi : G \to G'$ and $\psi : G' \to G''$ which are isomorphisms.

Therefore $\phi o \psi : G \to G''$ is an isomorphism and onto.

Thus $G \approx G'$ and $G' \approx G''$ $G \approx G'$ and $\Rightarrow G \approx G''$

i.e. relation $\approx$ is transitive.

Hence $\approx$ is equivalence relation.

Kernel of a homomorphism:

If f is a homomorphism of a group G into a group G' then a set K of all these elements of G which are mapped onto the identity e' of G' is called the Kernel of the homomorphism f i.e. if f is homomorphism of G into G', then K is the Kernel of f if

K = $\{ x \in G :$ f(x) = e' where e' is the identity of G' $\}$

Normal Subgroup:

A subgroup H of group G is called normal subgroup of G if for every $x \in G$ and for every $h \in H$, $xhx^{-1} \in H$.

From this definition, we conclude that H is a normal subgroup of G iff $xHx^{-1} \subseteq H \quad \forall x \in G$

Every group G have at least two normal subgroups, G itself and the subgroup consisting of the identity element 'e' alone, These are called improper normal subgroups.

Theorem: A subgroup H of a group G is normal if and only if $xHx^{-1} = H \quad \forall x \in G$

Proof: Let $xHx^{-1} = H \qquad \forall x \in G$

$$\Rightarrow xHx^{-1} \subseteq H \qquad \forall x \in G$$

H is a normal subgroup of G.

Conversely, let H is a normal subgroup of G.

$$\Rightarrow xHx^{-1} \subseteq H \qquad \forall x \in G \qquad (1)$$

Since $x \in G \Rightarrow x^{-1} \in G$

So $x^{-1}H(x^{-1})^{-1} \subseteq H \qquad \forall x \in G$

$\Rightarrow \quad x^{-1}Hx \subseteq H \qquad \forall x \in G$

$\Rightarrow \quad x\left(x^{-1}Hx\right)x^{-1} \subseteq xHx^{-1} \qquad \forall x \in G$

$\Rightarrow \quad H \subseteq xHx^{-1} \qquad \forall x \in G \qquad (2)$

with the help of equation (1) and (2) $xHx^{-1} = H \quad \forall x \in G$.

Theorem: The intersection of any two normal subgroup is a normal subgroup.

Proof: Let H and K be any two normal subgroups of a group G.

Since H and K are subgroups of G therefore $H \cap K$ is also subgroup of G.

Let $x \in G$ and $n \in H \cap K$

We have $n \in H \cap K$ i.e. $n \in H$ and $n \in K$

Since H is a normal subgroup of G,

Then for $x \in G$, $n \in H \Rightarrow xnx^{-1} \in H$

Similarly $x \in G$, $n \in K \Rightarrow xnx^{-1} \in K$

i.e. $xnx^{-1} \in H$ and $xnx^{-1} \in K \Rightarrow xnx^{-1} \in H \cap K$

i.e. for every $x \in G$, $xnx^{-1} \in H \cap K$ we have $xnx^{-1} \in H \cap K$

Hence $H \cap K$ is a normal subgroup of G.

Theorem: If N is a normal subgroup of G and H is any subgroup of G, then prove that NH is a normal subgroup of G.

Proof: Let n_1h_1 and be any two elements of NH

Then $n_1, n_2 \in N$ and $h_1, h_2 \in H$

To prove, NH is a subgroup of G, we should prove that $\left(n_1h_1\right)\left(n_2h_2\right)^{-1} \in NH$

$$\left(n_1h_1\right)\left(n_2h_2\right)^{-1} = n_1h_1h_2^{-1}n_2^{-1}$$

$$= n_1h_1h_2^{-1}n_2^{-1}.h_2h_1^{-1}h_1h_2^{-1}$$

$$= n_1\left[(h_1h_2^{-1})n_2^{-1}(h_1h_2^{-1})^{-1}\right](h_1h_2^{-1})$$

Now N is normal and $n_2^{-1} \in N$, $h_1h_2^{-1} \in G \Rightarrow (h_1h_2^{-1})n_2^{-1}(h_1h_2^{-1})^{-1} \in N$

Therefore $n_1\left[(h_1h_2^{-1})n_2^{-1}(h_1h_2^{-1})^{-1}\right] \in N$

Since H is a subgroup of G therefore

$$h_1 \in H \text{ , } h_2 \in H \quad \Rightarrow h_1h_2^{-1} \in NH$$

$$\Rightarrow n_1\left[(h_1h_2^{-1})n_2^{-1}(h_1h_2^{-1})^{-1}\right](h_1h_2^{-1}) \in NH$$

Hence NH is a normal subgroup of G.

Theorem: If f is a homomorphism of a group G into a group G' with kernel K, then K is a normal subgroup of G.

Proof: Let f be a homomorphism of a group G into a group G'. Let e, e' be the identities of G and G' respectively, If K be the kernel of f then

$$K = \{ \; x \in G : \; f(x) = e' \; \}$$

Since $f(e) = e'$, therefore $e \in K$. Thus K is not empty.

Let $a, b \in K$, then $ab^{-1} \in K$, $f(b) = e^{'}$

So $f(ab^{-1}) = f(a) f(b^{-1})$

$$= f(a)\{f(b)\}^{-1}$$

$$= e^{'}\{e^{-1}\}^{-1}$$

$$= e^{'}e^{'}$$

$$= e^{'}$$

i.e. $ab^{-1} \in K$ for $a, b \in K$

therefore K is a subgroup of G.

Let $g \in G$ and $k \in K$ then $f(k) = e^{'}$

So $f(gkg^{-1}) = f(g) f(k) f(g^{-1})$

$$= f(g)e^{'}\left(f(g)\right)^{-1}$$

$$= f(g)\left(f(g)\right)^{-1}$$

$$= e^{'}$$

i.e. $gkg^{-1} \in K$ for $g \in G$ and $k \in K$

Hence K is normal subgroup of G.

Questions:

(1) If $f : G \to \overline{G}$ be a group homomorphism, then under f identities and inverse corresponds i.e.

(i) $f(e) = \overline{e}$ where e and $\overline{e}$ are the identities of G and $\overline{G}$ respectively.

(ii) $f(a^{-1}) = \left[f(a)\right]^{-1}$ $\qquad \forall a \in G$

(2) The composition of two homomorphism is also homomorphism.

(3) The intersection of any collection of normal subgroups is itself a normal subgroup.

(4) Every subgroup of an abelian group is normal.

Reference :

1. Modern Algebra by A.R. Vashistha

2. Abstract Algebra by Khanna & Bhambri

3. Algebra & trigonometry by Pandey