Normal Distribution & It's Properties

Dr. Pawan Kumar
Assistant Professor
Department of Statistics
Udai Pratap College, Varanasi-221002
Email: pawanchauhanstranger@gmail.com

Introduction

- The normal distribution was first discovered in 1733 by English Mathematician De-Moivre, who obtained this distribution as a limiting case of binomial distribution and applied it to problems arises in the game of chance.
- It was also known to Laplace. no later than 1774 but through a historical error it was credited to Gauss. who first made reference to it in the beginning of 19^{th} century (1809) as the distributions of errors in Astronomy.
- Gauss used the normal curve to describe the theory of accidental errors of measurements involved in the calculation of orbits of heavenly bodies.
- Throughout the eighteenth and nineteenth centuries, various efforts were made to establish the normal model as the underlying law ruling all continuous random variables. Thus. the name **Normal**.
- The normal model has. nevertheless. become the most important probability model in statistical analysis.
- This is most widely used distribution in various aspects in different fields of society.
- This distribution is a finest suitable example of continuous distribution.

Form of the distribution

Univariate Normal distribution

A random variable X is said to have a normal distribution with parameters μ (called mean) and σ^2(called variance) if its density function is given by

$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left\{\left(\frac{x-\mu}{\sigma}\right)^2\right\}} \qquad x \in \mathbb{R}, \mu \in \mathbb{R} \;\; \& \;\; \sigma > 0. \tag{1}$$

- A random variable X following normal distribution with μ and σ^2 is usually denoted by $X \sim$N(μ, σ^2).
- If $X \sim$N(μ, σ^2) then $Z = \left(\frac{X-\mu}{\sigma}\right)$ is known as standard normal variate and follows $N(0,1)$. More symbolically, $Z \sim N(0,1)$

Properties of Normal distribution

Chief characteristics of Normal distribution & Normal Probability Curve

- Mean, median and Mode of the distribution coincides.
- Normal probability curve is bell shaped and curve is symmetrical about the line $X = \mu$.
- Linear combination of independent normal variates is also normal distribution.
- As x increases increasingly and $f(x)$ decreases rapidly and maximum probability occurring at the point $x = \mu$ and

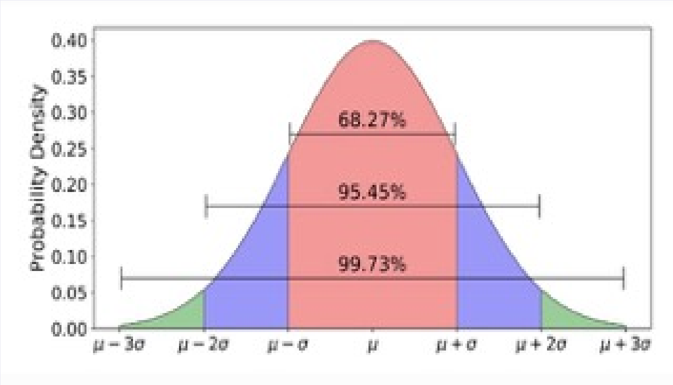
$$[f(x)]_{Max} = \frac{1}{\sigma\sqrt{2\pi}}$$

- Total x-axis is an asymptote to the curve and is at the point $x = \mu \pm \sigma$ and

$$[f(x)]_{x=\mu\pm\sigma} = \frac{1}{\sigma\sqrt{2\pi}}e^{-1/2}$$

Bell-shaped Curve

Normal Probability Curve



- Area Property
$$P(\mu - \sigma \leq X \leq \mu + \sigma) = 0.6826$$
$$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 0.9544$$
$$P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = 0.9973$$
- $$Q.D. : M.D. : S.D. = \frac{2}{3}\sigma : \frac{4}{5}\sigma : \sigma = 10 : 12 : 15$$
- $$\beta_1 = 0 \;\; \& \;\; \beta_2 = 3$$
- $$\mu_{2r+1} = 0$$
$$\mu_{2r} = (1.3.5...(2r-3)(2r-1))\,\sigma^{2r}; r = 0, 1, 2, \cdots$$
- If
$$X \sim N(\mu, \sigma^2) \implies \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$$

Forms of Bivariate & Multivariate Normal distribution

- If $X \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then

$$f(x; \mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} exp\left(-\frac{Q}{2}\right)$$

where

$$Q = \left[\frac{1}{(1-\rho^2)}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

- If $\boldsymbol{x} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then

$$f(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-p/2}|\boldsymbol{\Sigma}|^{-1/2} exp\left[(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]$$

Where, $\boldsymbol{\mu}$ is mean vector & $\boldsymbol{\Sigma}$ is the variance-covariance matrix.

Importance of Normal Distribution

- Most of the distributions occurring in practice, e.g., Binomial, Poisson, Hypergeometric, distributions. etc., can be approximated by normal distribution. Moreover, many of the sampling distributions. e.g., Student's 't' , Snedecor's F, Chi-square distributions, etc:, tend to normality for large samples.
- Many of the distributions of sample statistic (e.g., the distributions of sample mean, sample variance, etc.) tend to normality for large samples and as such they can best be studied with the help of the normal curves.
- The entire theory of small sample tests, viz., t, F, χ^2 tests etc., is based on the fundamental assumption that the parent populations from which the samples have been drawn follow normal distribution.
- Normal distribution finds large applications in Statistical Quality Control in industry for setting control limits.

Conclusion

Concluding Remarks

This presentation is based on well known normal distribution and its generalizations for bi-variate and multivariate case. Also, we have shown some existing properties of this distribution.

References

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Thank you ...