Last unit

Theory of moments, moment generating function and its properties.

Moments

Moments is used to know or identify the nature and behaviour of any data or distribution.

Types of Moments :-

(i) Moments about mean OR Central moment (μ_r)

(ii) Raw moments OR moments about any point OR Non-central moments (μ_r')

(iii) Factorial moments.

☆ Raw moments OR Moment about any assumed mean OR about any value

Definition :-

The r^{th} order raw moment about origin is denoted by μ_r' and defined as

$$\boxed{\mu_r' = E[x^r] = \begin{cases} \sum_x x^r \cdot P[X=x] & \text{Discrete} \\ \int_{-\infty}^{\infty} x^r \cdot f(x)\,dx & \text{Continuous} \end{cases} \quad ; \; r = 1, 2, 3, 4}$$

on putting $r = 1, 2, 3, 4$ we get

$$\mu_1' = \text{mean} = E(x) = \begin{cases} \sum_x x \cdot P[X=x] \\ \int_{-\infty}^{\infty} x \cdot f(x)\,dx \end{cases}$$

μ_2', μ_3' and μ_4'

μ_1' = first order Raw moment about origin

Central Moment :-

The r^{th} order moment about mean is denoted by μ_r and defined as

$$\boxed{\begin{aligned}\mu_r &= E(x-\bar{x})^r = E[x-E(x)]^r \\ &= \begin{cases} \sum_x (x-\bar{x})^r \cdot P[X=x] & ; \text{Discrete} \\ \int_{-\infty}^{\infty} (x-\bar{x})^r \cdot f(x)\,dx & ; \text{Continuous} \end{cases} \\ &\quad ; r = 1, 2, 3, 4\end{aligned}}$$

on putting $r = 1, 2, 3, 4$, we get

$$\mu_1 = E[x-\bar{x}]^1 = E[x-E(x)]$$

$$= E(x) - E(x) = 0$$

$$\Rightarrow \boxed{\mu_1 = 0}$$

This shows that first order central moment about mean is always zero.

$$\mu_2 = E(x-\bar{x})^2 = E(x-E(x))^2 \quad [\because E(x) = \bar{x}]$$

$$\Rightarrow \mu_2 = E[x - E(x)]^2 \ldots$$

$$\Rightarrow \boxed{\mu_2 = E[x - E(x)]^2 = \underline{E(x^2) - \{E(x)\}^2}}$$

This shows that second order central moment is variance.

$$\boxed{\begin{aligned} \mu_3 &= E[x - E(x)]^3 \\ \mu_4 &= E[x - E(x)]^4 \end{aligned}}$$

Rough

$$E[x - E(x)]^2 = E[x^2 + \{E(x)\}^2 - 2x E(x)]$$

$$= E(x^2) + \{E(x)\}^2 - 2E(x) \cdot E(x)$$

$$= E(x^2) + \{E(x)\}^2 - 2\{E(x)\}^2$$

$$= \underline{E(x^2) - \{E(x)\}^2}$$

Given that

$E(x^2) = 20, \quad E(x) = 4$

What is variance?

$$\text{Variance} = \mu_2 = E(x^2) - \{E(x)\}^2$$
$$= 20 - (4)^2$$
$$= 20 - 16 = 4$$

Variance = 4

$$\text{Standard deviation} = \sqrt{\text{variance}}$$
$$= \sqrt{4} = 2$$

$\Rightarrow$ $\boxed{\begin{array}{l}\text{variance} = 4 \\ \text{Standard Deviation} = 2\end{array}}$

Moment Generating Function :-

The moment generating function of random variable x at a real parameter 't' is denoted by $M_x(t)$ and defined as

$$M_x(t) = E(e^{tx})$$
$$= \begin{cases} \sum_x e^{tx} \cdot P[x=x] & \text{Discrete} \\ \end{cases}$$

$$\int_{-\infty}^{\infty} e^{tx} f(x)\,dx \quad \text{Continuous}$$

Properties :-

(1) If $X_1, X_2, X_3, \cdots, X_n$ are n independent random variables then

$$\boxed{M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)}$$

(2) Moment generating function is not independent of change of origin and scale

$$\boxed{M_{\left(\frac{X-\mu}{\sigma}\right)}(t) = e^{-\frac{\mu t}{\sigma}} M_X(t/\sigma)}$$