Module-I

Subject- Mathematics

Class & Year- B. Sc. I^{st} year

Topic: Confocal Conics

Keywords: Confocal Conics

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Module 1: Confocal Conics

1 Confocal Conics

Definition 1.1. *Two conics are said to be* ***confocal****, if they have the same foci.*

The ellipse and hyperbola have two foci each, which are on the principal axes at equal distances from the centre. Therefore, the confocal conics have common centre and common principal axes.

To find the equation to conics which are confocal with an given ellipse :
Let the equation of the given ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1.1}$$

The equation to any conic having the same centre and axes as the given ellipse is given by

$$\frac{x^2}{A} + \frac{y^2}{B} = 1 \tag{1.2}$$

The foci of (1.1) are $(\pm ae, 0) = (\pm\sqrt{a^2 - b^2}, 0)$, where $e = \frac{\sqrt{a^2-b^2}}{a}$ is the eccentricity of conic. The foci of (1.2) are $(\pm\sqrt{A - B}, 0)$. These foci are the same if $A - B = a^2 - b^2$, *i.e.*, if $A - a^2 = B - b^2 = \lambda$ (say) $\therefore A = a^2 + \lambda$, and $B = b^2 + \lambda$. The equation (1.2) then becomes $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$.

Hence, the equation of the system of conics confocal to (1.1) is

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \tag{1.3}$$

where λ is a parameter which determines a particular confocal.

1.1 Some Properties of Confocal Conics

(i) *Through any given point in the plane of a given conic, two confocals can be drawn, one of them is an ellipse and other is a hyperbola.*
Let the equation of the given conic be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let the given point in the plane conic be (h, k). Any conic confocal with the given conic is

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1. \tag{1.4}$$

If the conic (1.4) passes through the point (h, k), we have

$$\frac{h^2}{a^2 + \lambda} + \frac{k^2}{b^2 + \lambda} = 1. \tag{1.5}$$

Let us put $b^2 + \lambda = \mu$; then $a^2 + \lambda = a^2 + \mu - b^2 = a^2e^2 + \mu$. The equation (1.5) then becomes

$$\frac{h^2}{a^2e^2 + \mu} + \frac{k^2}{\mu} = 1,$$

which on simplifying gives

$$\mu^2 + (a^2e^2 - h^2 - k^2)\mu - a^2e^2k^2 = 0 \tag{1.6}$$

The discriminant of above quadratic (in μ) $= (a^2e^2 - h^2 - k^2)^2 + 4a^2e^2k^2 \geq 0$, which implies that the equation (1.6) has real roots, say μ_1 and μ_2. This gives two values of λ, say λ_1 and λ_2, corresponding to which we have two confocals given by (1.4) through the point (h, k).

Now, in equation (1.6), the product of roots $= \mu_1\mu_2 = -a^2e^2k^2 < 0$, which means one value of μ and the other is positive. The two values of $b^2 + \lambda$ are therefore, one positive and the other negative.

Again, let us put $a^2 + \lambda = \eta$; then $b^2 + \lambda = b^2 + \eta - a^2 = \eta - a^2e^2$. The equation (1.5) then gives

$$\eta^2 - (a^2e^2 + h^2 + k^2)\eta + a^2e^2h^2 = 0$$

Let η_1 and η_2 be the roots of the above equation, then $\eta_1 + \eta_2 = (a^2e^2 + h^2 + k^2) > 0$ and $\eta_1\eta_2 = a^2e^2h^2 > 0$. This implies that both the roots, η_1 and η_2, must be positive. Therefore, the two values of $a^2 + \lambda$ are positive.

On substituting in (1.4) we thus obtain two confocals, of which one is an ellipse and the other a hyperbola.

Alternative approach:
The equation (1.5) can be written as $\quad h^2(b^2 + \lambda) + k^2(a^2 + \lambda) = (a^2 + \lambda)(b^2 + \lambda)$.

$$f(\lambda) \equiv (a^2 + \lambda)(b^2 + \lambda) - h^2(b^2 + \lambda) - k^2(a^2 + \lambda) = 0$$

The above equation is a quadratic in λ, so there will be two values of λ. Therefore, there are two confocal conics which pass through the given point (h, k).

Now, we observe (assuming $a > b$) that

$$\lim_{\lambda \to -\infty} f(\lambda) = +\infty, f(-a^2) = -h^2(b^2 - a^2) > 0, f(-b^2) = -k^2(a^2 - b^2) < 0, \lim_{\lambda \to +\infty} f(\lambda) = +\infty$$

By intermediate value property, the equation $f(\lambda = 0)$, has two real roots λ_1 and λ_2 such that

$$-a^2 < \lambda_1 < -b^2 < \lambda_2.$$

When $\lambda = \lambda_1$, $a^2 + \lambda_1$ is positive but $b^2 + \lambda_1$ is negative, and hence the equation (1.4) represents a hyperbola. For $\lambda = \lambda_2$, both $a^2 + \lambda_2$ and $b^2 + \lambda_2$ are positive, therefore, the confocal (1.4) represents an ellipse.

(ii) *Two confocal conics cut each other at right angles.*
Let the two confocals conics are

$$\frac{x^2}{a^2 + \lambda_1} + \frac{y^2}{b^2 + \lambda_1} = 1, \quad \text{and} \quad \frac{x^2}{a^2 + \lambda_2} + \frac{y^2}{b^2 + \lambda_2} = 1,$$

and let they cut each other at the point (h, k).

Now, the equation of tangents to the confocals at (h, k) are

$$\frac{xh}{a^2 + \lambda_1} + \frac{yk}{b^2 + \lambda_1} = 1, \quad \text{and} \quad \frac{xh}{a^2 + \lambda_2} + \frac{yk}{b^2 + \lambda_2} = 1.$$

These tangents cut each other at right angles, if the product of their slopes is -1,

$$i.e., \text{ if} \qquad \left(-\tfrac{h}{a^2+\lambda_1}.\tfrac{b^2+\lambda_1}{k}\right) \times \left(-\tfrac{h}{a^2+\lambda_2}.\tfrac{b^2+\lambda_2}{k}\right) = -1,$$

$$i.e., \text{ if} \qquad \frac{h^2}{(a^2+\lambda_1)(a^2+\lambda_2)} + \frac{k^2}{(b^2+\lambda_1)(b^2+\lambda_2)} = 0. \tag{1.7}$$

But, since (h, k) is a common point of the two confocals, we have

$$\frac{h^2}{a^2+\lambda_1} + \frac{k^2}{b^2+\lambda_1} = 1, \quad \text{and} \quad \frac{h^2}{a^2+\lambda_2} + \frac{k^2}{b^2+\lambda_2} = 1,$$

Subtracting these, we have $(\lambda_1 - \lambda_2)\left[\frac{h^2}{(a^2+\lambda_1)(a^2+\lambda_2)} + \frac{k^2}{(b^2+\lambda_1)(b^2+\lambda_2)}\right] = 0$. But $\lambda_1 \neq \lambda_2$, we have

$$\frac{h^2}{(a^2+\lambda_1)(a^2+\lambda_2)} + \frac{k^2}{(b^2+\lambda_1)(b^2+\lambda_2)} = 0. \tag{1.8}$$

Since the condition in (1.7) is thus satisfied by the virtue of (1.8), confocals cut each other at right angle.

(iii) *One and only one conic, confocal with an ellipse, can be drawn to touch a given straight line.*

Let the given straight line be

$$lx + my = n. \tag{1.9}$$

The system of confocals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1, \tag{1.10}$$

where λ is a parameter. The equation of a tangent at (h, k) to (1.10) is

$$\frac{xh}{a^2+\lambda} + \frac{yk}{b^2+\lambda} = 1 \tag{1.11}$$

Now, the line (1.9) touches a member of (1.10) at (h, k) if the equations (1.9) and (1.11) are identical, that is,

$$\frac{l(a^2+\lambda)}{h} = \frac{m(b^2+\lambda)}{k} = n$$

which gives

$$h = \frac{l(a^2+\lambda)}{n}, \quad k = \frac{m(b^2+\lambda)}{n}$$

But the point (h, k) lies on conic (1.10), so we have

$$\frac{h^2}{a^2+\lambda} + \frac{k^2}{b^2+\lambda} = 1.$$

Substituting the values of h and k from above in the last equation, we get

$$l^2(a^2+\lambda) + m^2(b^2+\lambda) = n^2. \tag{1.12}$$

This is a linear equation in λ, so we get one and only value of λ. Hence only one conic from the system of confocal will touch the given line.

***Note: Equation** (1.12) **gives the condition that the straight line** (1.9) **will touch a conic of the form** (1.10).

(iv) *The difference of the squares of the perpendiculars drawn form the centre on any two parallel tangents to two given confocals is constant.*

Let the two given confocals be

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \quad \text{and} \quad \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1,$$

and the two parallel tangents to these confocals be $y = mx + c_1$ and $y = mx + c_2$, respectively.

By the condition of tangency in (1.12), we have

$$m^2(a^2+\lambda_1)+(b^2+\lambda_1)=c_1^2 \quad \text{and} \quad m^2(a^2+\lambda_2)+(b^2+\lambda_2)=c_2^2 \tag{1.13}$$

Now, if p_1 and p_2 be the lengths of perpendiculars from the centre to the tangents, then

$$p_1^2=\frac{c_1^2}{1+m^2} \quad \text{and} \quad p_2^2=\frac{c_2^2}{1+m^2} \tag{1.14}$$

Using (1.13) and (1.14), we get

$$p_1^2-p_2^2=\frac{c_1^2-c_2^2}{1+m^2}=\lambda_1-\lambda_2= \text{ a constant.} \tag{1.15}$$

(v) *The locus of a point from which two perpendicular tangents can be drawn one to each of two given confocals is a circle.*
Or, *The point of intersection of two perpendicular tangent lines one to each of two given confocal conics lies on a circle.*

Let the two given confocals be

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \quad \text{and} \quad \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1,$$

and the two perpendicular tangents to these confocals be $y = mx + c$ and $my + x = d$, respectively.

By the condition of tangency in (1.12), we have

$$m^2(a^2+\lambda_1)+(b^2+\lambda_1)=c^2 \quad \text{and} \quad (a^2+\lambda_2)+m^2(b^2+\lambda_2)=d^2. \tag{1.16}$$

Therefore, the equation of perpendicular tangents are

$$y=mx+\sqrt{m^2(a^2+\lambda_1)+(b^2+\lambda_1)} \quad \text{and} \quad my+x=\sqrt{(a^2+\lambda_2)+m^2(b^2+\lambda_2)}. \tag{1.17}$$

For the locus of the point of intersection of these tangent lines, we have to eliminate m between them. Squaring and adding these equations, we have

$$(y-mx)^2+(my+x)^2=m^2(a^2+\lambda_1)+(b^2+\lambda_1)+(a^2+\lambda_2)+m^2(b^2+\lambda_2)$$

which gives $\quad (1+m^2)(x^2+y^2)=(1+m^2)(a^2+b^2+\lambda_1+\lambda_2).$

Therefore, the required locus is $\quad x^2+y^2=a^2+b^2+\lambda_1+\lambda_2$, which is a circle.

(vi) The locus of the pole of a given straight line with respect to a system of confocals is a straight line.

Let the given line be

$$lx + my = n \tag{1.18}$$

and let the confocal be

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1. \tag{1.19}$$

If (h, k) be the pole of the line (1.18), with respect to the confocal (1.19), then the polar of (h, k) with respect to (1.19) is identical with (1.18).

Now polar of (h, k) with respect to (1.19) is

$$\frac{hx}{a^2+\lambda} + \frac{ky}{b^2+\lambda} = 1. \tag{1.20}$$

Comparing (1.18) and (1.20), we get

$$\frac{l(a^2+\lambda)}{h} = \frac{m(b^2+\lambda)}{k} = \frac{n}{1},$$

, which gives $a^2 + \lambda = \frac{nh}{l}, b^2 + \lambda = \frac{nk}{m}$.

Eliminating λ between these equations, we get $\frac{h}{l} - \frac{k}{m} = \frac{a^2-b^2}{n}$.

Therefore, the locus of the pole (h, k) of the given straight line is

$$\frac{x}{l} - \frac{y}{m} = \frac{a^2-b^2}{n},$$

which is a straight line perpendicular to the given line.

Solved Examples

1. Find the conic confocal with the conic $x^2 + 2y^2 = 2$, which passes through the point $(1, 1)$.
Sol.: The equation of the given conic can be written as

$$\frac{x^2}{2} + \frac{y^2}{1} = 1 \tag{1}$$

Then the equation of its confocal conic is given by

$$\frac{x^2}{2+\lambda} + \frac{y^2}{1+\lambda} = 1. \tag{2}$$

Since (2) has to pass through the given point $(1, 1)$, we have

$$\frac{1^2}{2+\lambda} + \frac{1^2}{1+\lambda} = 1$$

which on simplifying gives $\lambda^2 + \lambda - 1 = 0$, so $\lambda = \frac{-1\pm\sqrt{5}}{2}$.

Putting the values of λ above in (2), we get the required equation of confocal conics.

$$\frac{x^2}{2+\frac{-1\pm\sqrt{5}}{2}} + \frac{y^2}{1+\frac{-1\pm\sqrt{5}}{2}} = 1 \quad \text{or} \quad \frac{2x^2}{3\pm\sqrt{5}} + \frac{2y^2}{1\pm\sqrt{5}} = 1$$

On rationalizing the denominators and simplifying we get, $(3x^2 - y^2) \pm \sqrt{5}(y^2 - x^2) = 2$.

2. Show that the confocal hyperbola through the point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose eccentric angle is α, is $\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=a^2-b^2$.

Sol.: The equation of conic confocal to the given ellipse is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1. \tag{1}$$

Now we know that the co-ordinates of the point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having eccentric angle 'α' is $(a\cos\alpha, b\sin\alpha)$. Since the confocal (1) passes though the point $(a\cos\alpha, b\sin\alpha)$, we have

$$\frac{a^2\cos^2\alpha}{a^2+\lambda}+\frac{b^2\sin^2\alpha}{b^2+\lambda}=1$$

On simplifying, we get

$$\lambda^2+\lambda(a^2\sin^2\alpha+b^2\cos^2\alpha)=0$$

which gives $\lambda=0, -(a^2\sin^2\alpha+b^2\cos^2\alpha)$.

$\lambda=0$ gives the original ellipse itself. On putting the value of $\lambda=-(a^2\sin^2\alpha+b^2\cos^2\alpha)$ in (1), we get the required equation of confocal conic given by $\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=a^2-b^2$.

3. Show that the locus of the points of contact of the tangents through a fixed point (h,k) to the system of conics confocal with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is the cubic curve

$$\frac{x}{y-k}+\frac{y}{x-h}=\frac{a^2-b^2}{hy-kx}.$$

4. Prove that two conics $ax^2+2hxy+by^2=1$ and $a'x^2+2h'xy+b'y^2=1$ can be placed so as to be confocal if

$$\frac{(a-b)^2+4h^2}{(ab-h^2)^2}=\frac{(a'-b')^2+4h'^2}{(a'b'-h'^2)^2}.$$

Exercises

Q1. Find the conic confocal with the conic $x^2+9y^2=9$, which passes through the point $(3,1)$.

Q2. If the confocals through $P(x_1,y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1$ and $\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1$, then show that (i) $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1-\frac{\lambda_1\lambda_2}{a^2b^2}$, (ii) $x_1^2+y_1^2-a^2-b^2=\lambda_1+\lambda_2$. Also prove that the angle θ between the tangents from $P(x_1,y_1)$ to the ellipse is given by

$$\tan\ \theta=\frac{2\sqrt{-\lambda_1\lambda_2}}{\lambda_1+\lambda_2}.$$

Q3. Prove that the locus of the foot of normals drawn from a fixed point (h,k) on to the system of conics confocal with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is the cubic curve

$$\frac{x}{y-k}+\frac{y}{x-h}=\frac{a^2-b^2}{hy-kx}.$$

Q4. Prove that the locus of the points lying on a system of confocal ellipses which have the eccentric angle α, is a confocal hyperbola whose asymptotes are inclined at angle 2α.