Module-II

Subject- Mathematics

Class & Year- B. Sc. I^{st} year

Topic: Polar Equation of Conics

Keywords: Conics, Polar coordinates

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Module 2: Polar Equation of Conics

1 Polar Coordinates

In a plane, we can determine the position of a point P with respect to a fixed point O, called the ***pole***, and a fixed straight line OX, called the ***initial line***. We join P to O. If the length of OP is r and $\angle XOP$ (which is traced out by the line OP in revolving from the initial line OX) is θ, then the ***polar coordinates*** of P is given by (r, θ). Here, r is called the ***radius vector*** and θ is called the ***vectorial angle*** of the point P (see Figure 1).

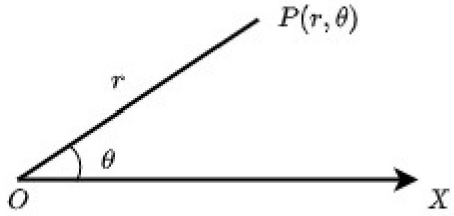


Figure 1: Polar Coordinate

The radius vector r is positive if it is measured from the origin along the line bounding the vectorial angle; if measured in the opposite direction it is negative. The vectorial angle θ is positive, if it is measured in anti-clockwise direction from OX and it is negative, if is measured in clockwise direction from OX.

The polar coordinates of a point are not unique as in the Cartesian coordinates (see Figure 2). Adding 2π (or any multiple of 2π) to the vectorial angle does not alter the final position of the revolving line, so (r, θ) is always same point as $(r, \theta + 2n\pi)$, where n is an integer. Also, adding π or any odd multiple of π to the vectorial angle and changing the sign of radius vector gives the same point as before. Thus, the point $(-r, \theta + (2n+1)\pi)$ is the same point as $(-r, \theta + \pi)$, *i.e.*, is the point (r, θ).

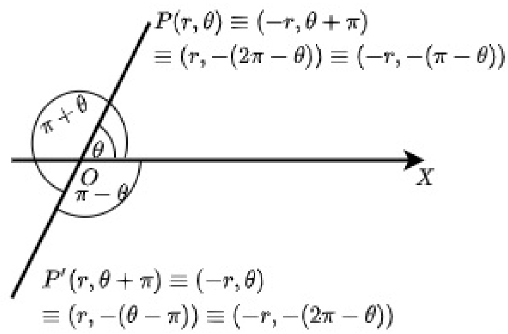


Figure 2: Different representations of polar coordinates of a point

If PO is produced to P' so that $OP = OP'$ in magnitude, then the the coordinates of P' are either $(r, \theta + \pi)$ or $(-r, \theta)$.

In general, taking into consideration the signs of polar coordinates (as discussed above), it is easy to see that the same point is represented by each of the following polar coordinates:
$(r, \theta), (-r, \theta + \pi), (r, -(2\pi - \theta)), (-r, -(\pi - \theta))$.

Now, we are going to present some results regarding polar coordinates which would be used in upcoming sections.

(i) ***Relation between Cartesian and Polar Coordinates:***

Let P be any point whose Cartesian coordinates, referred to rectangular axes, are (x, y), and whose polar coordinates, referred to O as pole and OX as initial line, are (r, θ). Then, $OP = r$ and $\angle MOP = \theta$.

Draw PM perpendicular from P to OX, so that we have $OM = x, MP = y$ (see Figure 3).

From $\triangle MOP$, we have

$$\frac{PM}{OP} = \sin\theta \quad \text{and} \quad \frac{OM}{OP} = \cos\theta$$

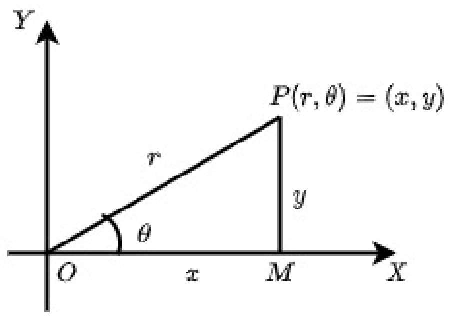


Figure 3: Cartesian and Polar Coordinate

$$\text{or} \quad \frac{y}{r} = \sin\theta \quad \text{and} \quad \frac{x}{r} = \cos\theta$$

which gives

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{1.1}$$

and hence, $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$, that is,

$$r = \sqrt{x^2+y^2} \quad \text{and} \quad \theta = \arctan\left(\frac{y}{x}\right). \tag{1.2}$$

Using equations (1.1) and (1.2), we can transform the polar coordinates into the Cartesian coordinates and vice versa.

(ii) *Distance between two points whose polar coordinates are given:*

Let P_1 and P_2 be two given points and let their polar coordinates be (r_1, θ_1) and (r_2, θ_2) respectively, so that, $OP_1 = r_1, OP_2 = r_2, \angle XOP_1 = \theta_1$ and $\angle XOP_2 = \theta_2$, where O is the pole and OX is the initial line. Then $\angle P_1OP_2 = \theta_2 - \theta_1$.

Using cosine rule in the triangle OP_1P_2 we have (see Figure 4)

$$\cos\angle P_1OP_2 = \frac{OP_1^2 + OP_2^2 - P_1P_2^2}{2OP_1 \cdot OP_2} = \frac{r_1^2 + r_2^2 - P_1P_2^2}{2r_1r_2},$$

which gives $\quad P_1P_2^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)$.

Therefore, the required distance between the points is given by

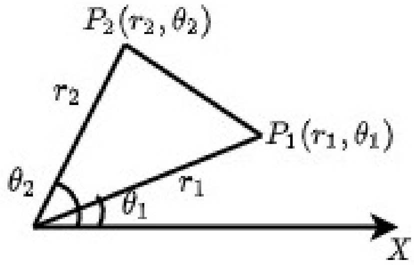


Figure 4: Distance between two given points

$$P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)} \tag{1.3}$$

(iii) *Polar equation of a straight line:*

We know that the general equation of a straight line in rectangular Cartesian coordinate system can be written as

$$ax + by = l.$$

Taking the origin as the pole and the positive x-axis as the initial line, the above equation, in polar coordinates (using transformation equation (1.1)), becomes

$$\frac{l}{r} = a\cos\theta + b\sin\theta. \tag{1.4}$$

If the straight line passes through the pole, then $l = 0$ and (1.4) becomes $a\cos\theta + b\sin\theta = 0$.

Above equation (1.4) is called the general equation of a straight line in polar coordinates. It can also be written as

$$r\cos(\theta - \alpha) = p,$$

where $\cos\alpha = a/\sqrt{a^2+b^2}, \sin\alpha = b/\sqrt{a^2+b^2}$ and $p = l/\sqrt{a^2+b^2}$.

Parallel and perpendicular lines: Since $ax + by = l$ and $ax + by = l'$ represent parallel lines, therefore the lines

$$\frac{l}{r} = a\cos\theta + b\sin\theta \quad \text{and} \quad \frac{l'}{r} = a\cos\theta + b\sin\theta$$

represent parallel lines in polar coordinates.

Again, since the line $bx - ay = l'$ is perpendicular to the line $ax + by = l$, the line perpendicular to the line

$$\frac{l}{r} = a\cos\theta + b\sin\theta$$

is given by

$$\frac{l'}{r} = b\cos\theta - a\sin\theta \quad \text{or} \quad \frac{l'}{r} = a\cos\left(\theta + \frac{\pi}{2}\right) + b\sin\left(\theta + \frac{\pi}{2}\right).$$

Therefore, the equation of any line perpendicular to the line (1.4) is obtained by writing $\left(\theta + \frac{\pi}{2}\right)$ for θ and changing l to a new constant L.

(iv) *Polar Equation of a straight line in normal form:*

Let p be the length of the perpendicular OM from the origin to the straight line, and let α be the angle which this perpendicular makes with the initial line OX as shown in Figure 6. Let $P(r, \theta)$ be any point on the required line. Then $OP = r, \angle XOP = \theta$. In triangle $\triangle OPM$,

$$OP = r, \angle POM = \alpha - \theta, OM = p.$$

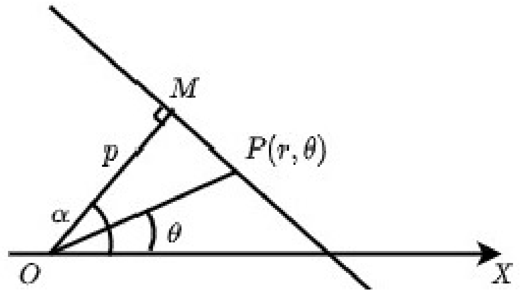


Figure 5: Straight line in normal form

Now,

$$\cos\angle POM = \frac{OM}{OP} \quad \text{or,} \ \cos(\alpha - \theta) = \frac{p}{r},$$

which gives $\cos(\theta - \alpha) = \frac{p}{r}$.

Therefore, the required equation of straight line is

$$r\cos(\theta - \alpha) = p. \tag{1.5}$$

Particular Cases:

Case (i). If the straight passes through the pole, then $p = 0$ and the equation (1.5) reduces to $r\cos(\theta - \alpha) = 0$ or, $\theta =$ constant.

Case (ii). If the straight line is perpendicular to the initial line, then $\alpha = 0$ and the equation of line reduces to $r\cos\theta = p$.

Case (iii). If the straight line is parallel to the initial line and is above it, then $\alpha = \pi/2$ and the equation of line reduces to $r\sin\theta = p$.

Case (iv). If the straight line is parallel to the initial line and is below it, then $\alpha = -\pi/2$ or $3\pi/2$ and the equation of line reduces to $r\sin\theta = -p$.

(v) *Area of the triangle whose vertices have polar coordinates* $(r_i, \theta_i), i = 1, 2, 3$.

Let $P_1P_2P_3$ be a triangle whose vertices have polar coordinates $(r_i, \theta_i), i = 1, 2, 3$ taken in order as shown in Figure 6. Then, we have $OP_1 = r_1, OP_2 = r_2, OP_3 = r_3$ and $\angle XOP_1 = \theta_1, \angle XOP_2 = \theta_2, \angle XOP_3 = \theta_3$.

$\therefore\ \angle P_1OP_2 = \theta_2 - \theta_1, \angle P_2OP_3 = \theta_3 - \theta_2$ and $\angle P_1OP_3 = \theta_3 - \theta_1$.

Now, we have (see Figure 6)

$$\text{area of } \triangle P_1P_2P_3 = \text{ area of } \triangle OP_1P_2 + \text{ area of } \triangle OP_2P_3 - \text{area of } \triangle OP_1P_3$$

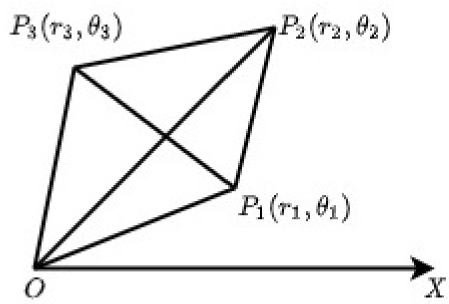


Figure 6: Area of a triangle with given vertices

But,

$$\begin{aligned}\text{area of } \triangle OP_1P_2 &= \frac{1}{2}OP_1 \cdot OP_2 \sin P_1OP_2 \\ &= \frac{1}{2}r_1r_2\sin(\theta_2 - \theta_1).\end{aligned}$$

Similarly, area of $\triangle OP_2P_3 = \frac{1}{2}r_2r_3\sin(\theta_3-\theta_2)$ and area of $\triangle OP_1P_3 = \frac{1}{2}r_1r_3\sin(\theta_3-\theta_1)$. Therefore,

$$\begin{aligned}\text{area of } \triangle P_1P_2P_3 &= \frac{1}{2}r_1r_2\sin(\theta_2-\theta_1)+\frac{1}{2}r_2r_3\sin(\theta_3-\theta_2)-\frac{1}{2}r_1r_3\sin(\theta_3-\theta_1)\\ &= \frac{1}{2}\left(r_1r_2\sin(\theta_2-\theta_1)+r_2r_3\sin(\theta_3-\theta_2)+r_3r_1\sin(\theta_1-\theta_3)\right). \quad (1.6)\end{aligned}$$

(vi) ***Polar Equation of a straight line passing through two given points:***

Let $A(r_1,\theta_1)$ and $B(r_2,\theta_2)$ be two given points and $P(r,\theta)$ be any point on the line joining the points A and B.

Now, the three points A, P and B lie on a straight line, the area of the triangle APB is zero (see Figure 7).

$\therefore \quad r_1r_2\sin(\theta_2-\theta_1)+rr_1\sin(\theta_1-\theta)+rr_2\sin(\theta-\theta_2)=0$ (using equation (1.6))

$\implies r_1r_2\sin(\theta_2-\theta_1)=rr_1\sin(\theta-\theta_1)+rr_2\sin(\theta_2-\theta)$.

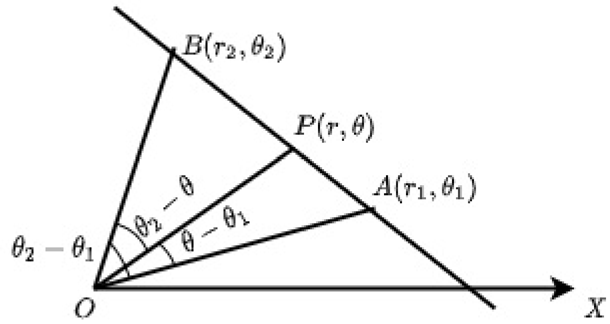


Figure 7: Straight line through two given points

Dividing both sides by rr_1r_2, we get

$$\frac{\sin(\theta_2-\theta_1)}{r}=\frac{\sin(\theta-\theta_1)}{r_2}+\frac{\sin(\theta_2-\theta)}{r_1} \quad (1.7)$$

Equation (1.7) is the required equation of the straight line. One might reach to same result by noting that, area of $\triangle AOB =$ area of $\triangle AOP +$ area of $\triangle POB$.

(vii) ***Polar Equation of a Circle:***

Let the radius of the circle be a and $C(c,\alpha)$ be its center. Take an arbitrary point $P(r,\theta)$ on the circle. Then we have $OC=c, \angle XOC=\alpha, OP=r, \angle XOP=\theta$.

$\therefore \quad \angle POC=\theta-\alpha$.

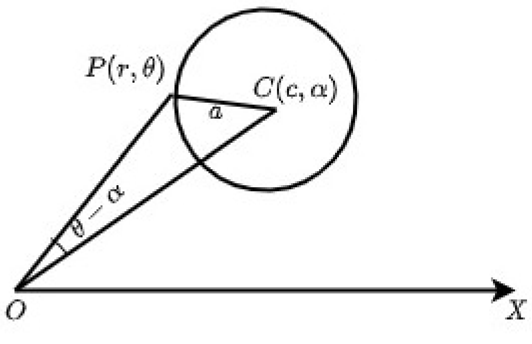


Figure 8: Circle with center (C,α) and radius a

Using cosine formula in $\triangle OPC$, we have

$$\begin{aligned} \cos\angle POC &= \frac{OP^2 + OC^2 - PC^2}{2OP\cdot OC} \\ \cos(\theta-\alpha) &= \frac{r^2+c^2-a^2}{2rc} \end{aligned}$$

Therefore, the equation of required circle is

$$\boxed{r^2 + c^2 - 2rc\cos(\theta-\alpha) = a^2.} \tag{1.8}$$

Particular cases:
Case (i). If the circle passes through the pole, then we have $c = a$ and the equation (1.8) reduces to

$$r = 2a\cos(\theta-\alpha).$$

Case (ii). If the center of the circle lies on the initial line, then we have $\alpha = 0$ and the equation (1.8) reduces to

$$r^2 - 2rc\cos\theta + c^2 - a^2 = 0.$$

Case (iii). If the pole be on the circle and also the initial line pass through the center of the circle, then $c = a$ and $\alpha = 0$. In this case, the equation of circle reduces to

$$r = 2a\cos\theta.$$

2 Polar Equation of Conics

2.1 Polar Equation of a Conic

(1) *To find the polar equation of a conic with its latus rectum of length $2l$, eccentricity e and the focus being the pole:*

Let S be the focus of the conic which is being taken as the pole. Let ZM be the directrix and A be the vertex of the conic. Draw a perpendicular SZ from S to the directrix, and take SZ (axis of conic) as the positive direction of the initial line.

Let SLS' be the latus rectum of length $2l$ so that the semi-latus rectum $SL = l$, and let e be the eccentricity of the conic.

Take any point $P(r,\theta)$ on the conic. Then $SP = r$ and $\angle XSP = \theta$.

Draw perpendiculars PM and LT to the directrix from P and L, respectively. Also, draw perpendicular PN from P to the initial line (axis). Then, we have $PM = NZ$ and $LT = SZ$ (see Figure 9).

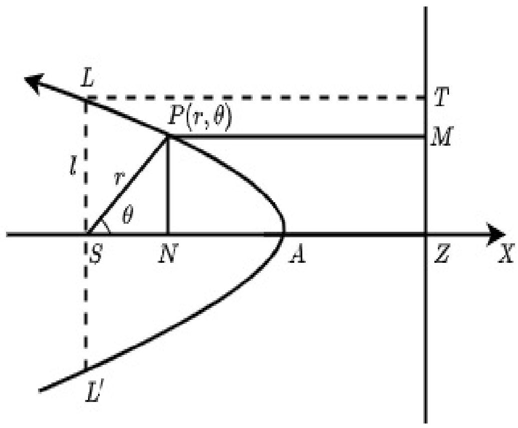


Figure 9: Conic: $\frac{l}{r} = 1 + e\cos\theta$

Since the points P and L are on the conic, therefore by the definition of a conic, we have

$$SP = e\cdot PM \quad \text{and} \quad SL = e\cdot LT \tag{2.1}$$

From the first relation in above equation, we have

$$\begin{aligned} SP &= e\cdot NZ = e\cdot(SZ - SN) \\ &= e\cdot SZ - e\cdot SP\cos\theta \\ \text{or} \qquad r &= e\cdot SZ - er\cos\theta \end{aligned}$$

which gives

$$r(1 + e\cos\theta) = e \cdot SZ \tag{2.2}$$

But using second relation in equation (2.1), we have

$$SL = e \cdot SZ \quad \text{or} \quad l = e \cdot SZ$$

Substituting this in (2.1), we get

$$r(1 + e\cos\theta) = l$$

Hence the polar equation of the conic is

$$\boxed{\frac{l}{r} = 1 + e\cos\theta.} \tag{2.3}$$

If the distance SZ between the focus S and the fixed line ZM is d, then $l = ed$ and the above equation of the conic can be written as

$$\frac{ed}{r} = 1 + e\cos\theta.$$

Remark 1: In the above derivation, we have taken SZ (the direction directed from the focus towards the directrix) as the positive direction of the initial line. However, if we take the positive direction of the initial line as ZS, that is, opposite to the direction directed from the focus towards the directrix (see Figure 10), then the equation of the conic comes out to be

$$\frac{l}{r} = 1 - e\cos\theta \tag{2.4}$$

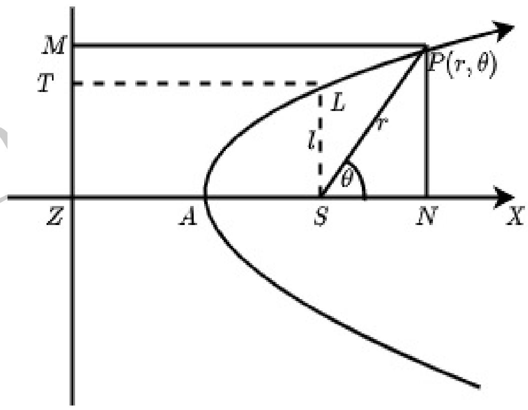


Figure 10: Conic: $\frac{l}{r} = 1 - e\cos\theta$

In this case,

$$\begin{aligned} SP &= e \cdot MP = e \cdot ZN = e \cdot (ZS + SN) \\ &= e \cdot (TL + SP\cos\theta) = e \cdot LT + e \cdot SP\cos\theta \end{aligned}$$

This gives

$$r = l + er\cos\theta \quad \text{or} \quad \frac{l}{r} = 1 - e\cos\theta.$$

Corollary 1: If the conic is a parabola with latus rectum $4a$, then $e = 1$ and $l = 2a$, and the equation (2.3) takes the form

$$\frac{2a}{r} = 1 + \cos\theta = 2\cos^2\frac{\theta}{2} \quad \text{or} \quad r = a\sec^2\frac{\theta}{2}. \tag{2.5}$$

Corollary 2: If α be the vectorial angle of a point A on conic $\frac{l}{r} = 1 + e\cos\theta$, then the radius vector of A, say r_A, will satisfy $\frac{l}{r_A} = 1 + e\cos\alpha$, which gives $r_A = \frac{l}{1+e\cos\alpha}$, and therefore, the polar coordinates of point A may be written as $\left(\frac{l}{1+e\cos\alpha}, \alpha\right)$. This is also called the point $'\alpha'$.

(2) *To find the polar equation of a conic with its focus being the pole and its axis inclined at an angle α to the initial line:*

Let S be the focus coinciding with the pole and let SZ, the axis of the conic, be inclined at an angle α to the initial line SX.

Consider a point $P(r,\theta)$ on the conic. Draw perpendiculars PM and PN from P on the directrix ZM and the axis SZ respectively.

Let $SL(=l)$ be the semi-latus rectum of the conic and $LT(=SZ)$ be the perpendicular from L to the directrix ZM (see Figure 11).

By the definition of a conic, we have

$$\begin{aligned} SP &= e\cdot PM \\ &= e\cdot NZ \quad (\because PM = NZ) \\ &= e\cdot(SZ-SN) = e\cdot SZ - e\cdot SN \\ &= e\cdot LT - e\cdot SP\cos(\theta-\alpha) \\ &\quad (\because SZ = LT \text{ and } SN = SP\cos(\theta-\alpha)) \\ \text{or} \quad & SP\cdot(1+e\cos(\theta-\alpha)) = e\cdot LT = SL \end{aligned}$$

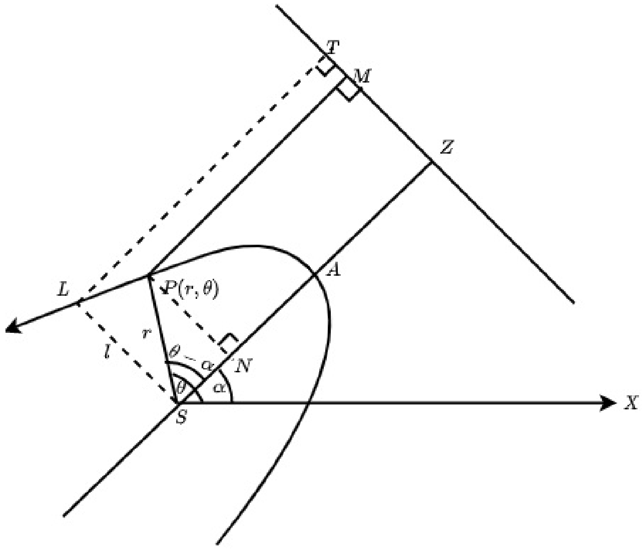


Figure 11: Conic: $\frac{l}{r} = 1 + e\cos(\theta-\alpha)$

This gives the required equation of conic as

$$r\,(1+e\cos(\theta-\alpha)) = l \quad \text{or} \quad \boxed{\frac{l}{r} = 1 + e\cos(\theta-\alpha),} \tag{2.6}$$

Note: Any result for the conic (2.6) can be obtained from the corresponding result for the conic (2.3) by replacing θ by $\theta-\alpha$., i.e., by writing $\theta-\alpha$ for θ, $\beta-\alpha$ for β etc.

Particular Cases:

Case 1. If the axis SZ of the conic coincides with the initial line (that is, $\alpha = 0$), then its equation becomes $\frac{l}{r} = 1 + e\cos\theta$, which we have derived earlier as equation (2.3).

Case 2. If the positive direction of the axis is opposite to the that of the initial line (that is, $\alpha = \pi$), then the equation of the conic becomes $\frac{l}{r} = 1 - e\cos\theta$, (see Remark 1).

2.2 Equations of the Directrices of a conic

To find the polar equation of the directrices of the conic $\frac{l}{r} = 1 + e\cos\theta$.

Case 1. When conic is an ellipse:

Let S be the focus coinciding with the pole and ZM be the directrix of ellipse corresponding to the focus S. Let $P(r,\theta)$ be any point on the directrix ZM. Then $SP = r$ and $\angle ZSP = \theta$.

From $\triangle PSZ$ (see Figure 12(a)), we have

$$\frac{SZ}{SP} = \cos\theta \text{ or } SZ = SP\cos\theta = r\cos\theta$$

But, L being a point on ellipse, we have

$$SL = e\cdot LT = e\cdot SZ \text{ or } SZ = \frac{SL}{e} = \frac{l}{e}$$

Therefore,

$$r\cos\theta = \frac{l}{e} \quad \text{or} \quad \frac{l}{r} = e\cos\theta, \tag{2.7}$$

which is the required equation of the directrix ZM.

To obtain the equation of other directrix $Z'M'$ corresponding to the focus S' (other than the pole S), we consider any point $P'(r', \theta')$ on it. Then $SP' = r'$ and $\angle ZSP' = \theta'$, and therefore, $\angle Z'SP' = \pi - \theta'$.

From $\triangle P'SZ'$,

$$SZ' = SP' \cos(\pi - \theta') \qquad \text{or} \quad SZ' = -r' \cos\theta'. \tag{*}$$

But in case of an ellipse

$$l = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2)$$

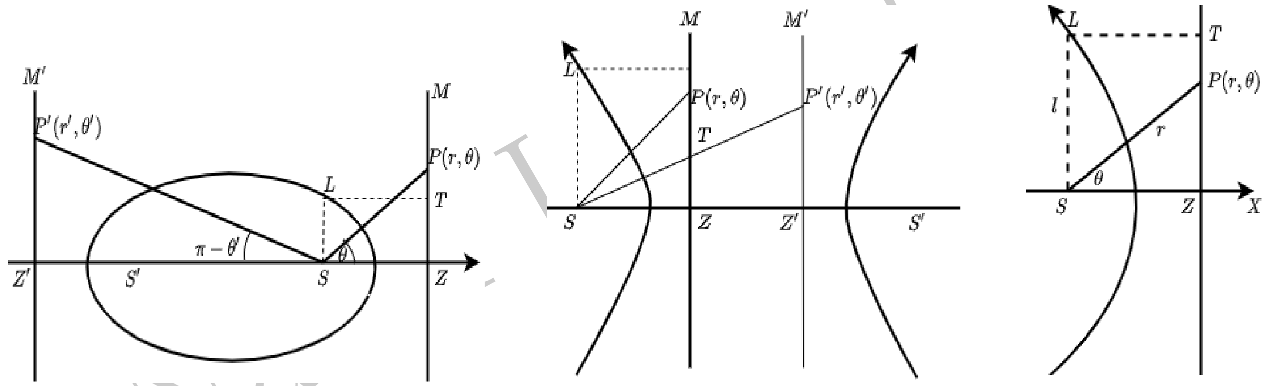
and therefore

$$SZ' = ZZ' - SZ = \frac{2a}{e} - \frac{l}{e} = \frac{2l}{e(1-e^2)} - \frac{l}{e} = -\left(\frac{e^2+1}{e^2-1}\right)\frac{l}{e}.$$

Putting this value of SZ' in (*) above, we get

$$r\cos\theta = \left(\frac{e^2+1}{e^2-1}\right)\frac{l}{e} \quad \text{or} \quad \frac{l}{r} = \left(\frac{e^2-1}{e^2+1}\right) e\cos\theta, \tag{2.8}$$

which is the required equation of the directrix $Z'M'$.



(a) Directrices of an ellipse $\frac{l}{r} = 1 + e\cos\theta$. (b) Directrices of a hyperbola $\frac{l}{r} = 1 + e\cos\theta$. (c) Directrix of a parabola $\frac{l}{r} = 1 + \cos\theta$.

Figure 12: Directrices of a conic.

Case 2. When conic is a hyperbola:

Let ZM and $Z'M'$ be the directrices of hyperbola corresponding to foci S (which is also the pole) and S' respectively. Proceeding in same manner as described in Case 1 above, the equation of ZM is obtained and is given by

$$\frac{l}{r} = e\cos\theta \tag{1}$$

For the equation of directrix $Z'M'$, take $P'(r', \theta')$ on it. Then $SP' = r', \angle Z'SP' = \theta'$.

From $\triangle P'SZ'$ (see Figure 12(b)),

$$SZ' = SP' \cos\theta' \text{or} \quad SZ' = r'\cos\theta'. \tag{**}$$

But, in case of a hyperbola

$$l = \frac{b^2}{a} = \frac{a^2(e^2-1)}{a} = a(e^2-1)$$

and therefore

$$\begin{aligned} SZ' &= SZ + ZZ' = \frac{l}{e} + \frac{2a}{e} \\ &= \frac{l}{e} + \frac{2l}{e(e^2-1)} = \left(\frac{e^2+1}{e^2-1}\right)\frac{l}{e}. \end{aligned}$$

Putting this value of SZ' in (**) above, we get

$$r\cos\theta = \left(\frac{e^2+1}{e^2-1}\right)\frac{l}{e} \quad \text{or} \quad \frac{l}{r} = \left(\frac{e^2-1}{e^2+1}\right)e\cos\theta, \tag{2}$$

which is the required equation of the directrix $Z'M'$.
(Note that the equations of directrices in case of ellipse and hyperbola are same.)

Case 3. When conic is a parabola:
In this case, we have a unique focus S and only one directrix ZM. Let $P(r,\theta)$ be a point on ZM. Then $SP = r$ and $\angle ZSP = \theta$. We have, $SZ = SP\cos\theta = r\cos\theta$. Also, $SZ = LT = SL = l$. This gives, $r\cos\theta = l$ and therefore the equation of directrix ZM is given by

$$\frac{l}{r} = \cos\theta.$$

Solved Examples

1: Show that the equations $\frac{l}{r} = 1 + e\cos\theta$ and $\frac{l}{r} = -1 + e\cos\theta$ represent the same conic.
Sol.: The equations of given conics are

$$\frac{l}{r} = 1 + e\cos\theta \tag{1}$$

$$\text{and} \quad \frac{l}{r} = -1 + e\cos\theta. \tag{2}$$

Let $P(r_1,\theta_1)$ be a point on the conic (1). Since the point (r_1,θ_1) can also be written as $(-r_1, \pi+\theta_1)$, therefore it also lies on (1) and so, we have

$$\begin{aligned} \frac{l}{-r_1} &= 1 + e\cos(\pi+\theta_1) \\ \Longrightarrow -\frac{l}{r_1} &= 1 - e\cos\theta_1 \Longrightarrow \frac{l}{r_1} = -1 + e\cos\theta_1, \end{aligned}$$

which shows the point $P(r_1,\theta_1)$ lies on the conic (2).
Again, if (r',θ') is a point on (2), then

$$\begin{aligned} \frac{l}{r'} &= -1 + e\cos\theta' \\ \Longrightarrow -\frac{l}{r'} &= 1 - e\cos\theta' \Longrightarrow \frac{l}{-r'} = 1 + e\cos(\pi+\theta') \end{aligned}$$

which shows that the point $(-r', \pi+\theta')$ lies on conic (1). But the point (r',θ') is same as the point $(-r', \pi+\theta')$. So, (r',θ') also lies on conic (1).

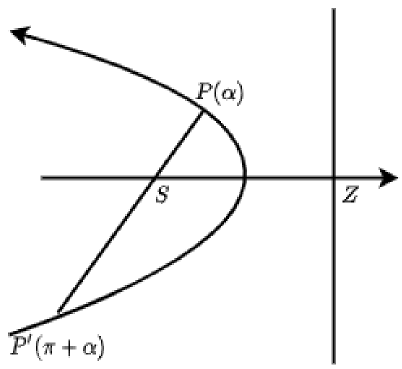
Thus, we have shown that each point on conic (1) also lies on conic (2), and vice versa, and therefore, conics represented by (1) and (2) are same.

2: In a conic, prove the following:
(i) the sum of the reciprocals of the segments of any focal chord of conic is constant.
(ii) the sum of the reciprocals of two perpendicular focal chords is constant.

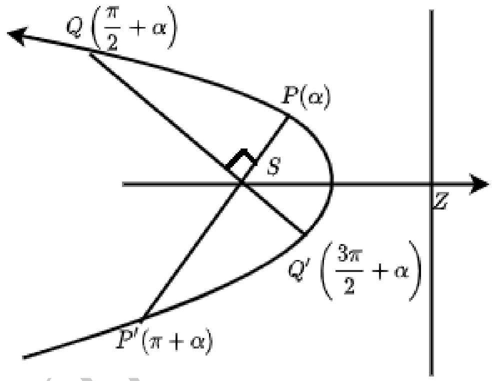
Sol.: Let the equation of the conic be

$$\frac{l}{r} = 1 + e\cos\theta. \tag{1}$$

(i) Let PSP' be any focal chord of conic such that the vectorial angle of P is α. Then the vectorial angle of P' is $\pi + \alpha$. Thus the polar coordinates of P and P' are (SP, α) and $(SP', \pi + \alpha)$ respectively (see Figure (a) below).



(a) Focal chord PSP' of conic



(b) Perpendicular focal chords PSP' and QSQ' of conic

Since P and P' both lie on the conic, we have

$$\frac{l}{SP} = 1 + e\cos\alpha \text{ and } \frac{l}{SP'} = 1 + e\cos(\pi + \alpha) = 1 - e\cos\alpha.$$

This gives

$$\frac{l}{SP} + \frac{l}{SP'} = 2 \quad \text{or} \quad \frac{1}{SP} + \frac{1}{SP'} = \frac{2}{l} = \text{a constant},$$

which implies that the sum of the reciprocals of the segments of any focal chord of conic is constant. The above result can also be expressed as "*the semi-latus rectum is the harmonic mean between the segments of a focal chord.*"
(ii) Let PSP' and QSQ' be two perpendicular focal chords of conic (see Figure (b) above). Then the vectorial angles of Q, P' and Q' are $\frac{\pi}{2} + \alpha, \pi + \alpha$ and $\frac{3\pi}{2} + \alpha$ respectively.

Since P, Q, P' and Q' lie on conic, we have

$$\begin{aligned}\frac{l}{SP} &= 1 + e\cos\alpha, \frac{l}{SQ} = 1 + e\cos\left(\frac{\pi}{2} + \alpha\right) = 1 - e\sin\alpha, \\ \frac{l}{SP'} &= 1 + e\cos(\pi + \alpha) = 1 - e\cos\alpha, \frac{l}{SQ'} = 1 + e\cos\left(\frac{3\pi}{2} + \alpha\right) = 1 + e\sin\alpha\end{aligned}$$

This gives

$$SP = \frac{l}{1 + e\cos\alpha}, \ SQ = \frac{l}{1 - e\sin\alpha}, \ SP' = \frac{l}{1 - e\cos\alpha}, \ SQ' = \frac{l}{1 + e\sin\alpha}$$

Therefore,

$$PP' = SP + SP' = \frac{l}{1 + e\cos\alpha} + \frac{l}{1 - e\cos\alpha} = \frac{2l}{1 - e^2\cos^2\alpha}$$
$$QQ' = SQ + SQ' = \frac{l}{1 - e\sin\alpha} + \frac{l}{1 + e\sin\alpha} = \frac{2l}{1 - e^2\sin^2\alpha}$$

which shows that

$$\frac{1}{PP'} + \frac{1}{QQ'} = \frac{1 - e^2\cos^2\alpha}{2l} + \frac{1 - e^2\sin^2\alpha}{2l} = \frac{2 - e^2}{2l} = \text{ a constant.}$$

3: Prove that the locus of middle points of focal chords of a conic is a conic of the same type.
Sol.: Let PSQ be a focal chord of the conic

$$\frac{l}{r} = 1 + e\cos\theta, \tag{1}$$

and α be the vectorial angle of P, then the vectorial angle of Q will be $\pi + \alpha$.

Since P and Q lie on the conic (1), we have

$$\frac{l}{SP} = 1 + e\cos\alpha \text{ and } \frac{l}{SQ} = 1 + e\cos(\pi + \alpha) = 1 - e\cos\alpha. \tag{2}$$

Let $R(r', \theta')$ be the middle point of the focal chord PSQ, then from Figure (13), $\theta' = \alpha$ and

$$\begin{aligned} r' &= SR = SP - PR = SP - \frac{SP + SQ}{2} = \frac{SP - SQ}{2} \\ &= \frac{1}{2}\left[\frac{l}{1 + e\cos\alpha} - \frac{l}{1 - e\cos\alpha}\right] = -\frac{le\cos\alpha}{1 - e^2\cos^2\alpha}. \quad \text{(using (2))} \end{aligned}$$

Therefore,

$$r' = -\frac{le\cos\theta'}{1 - e^2\cos^2\theta'}.$$

Hence, the locus of $R(r', \theta')$ is the curve

$$r = -\frac{le\cos\theta}{1 - e^2\cos^2\theta}, \ \textit{i.e.,}\ r^2(1 - e^2\cos^2\theta) + ler\cos\theta = 0.$$

Transforming the above equation to Cartesian coordinates, we have

$$x^2 + y^2 - e^2x^2 + lex = 0 \text{ or } (1 - e^2)x^2 + y^2 + lex = 0, \tag{3}$$

which being a second degree equation in x and y represents a conic.

The equation (3) represents an ellipse, a parabola or a hyperbola according as

$$0^2 - (1 - e^2) <, = \text{ or } > 0, \quad \text{that is, } e <, = \text{ or } > 1.$$

Thus, the conic (3) is of the same type as the given conic (1).

4: If PSQ and $PS'R$ be two chords of an ellipse through the foci S and S', then prove that $\frac{PS}{SQ} + \frac{PS'}{S'R}$ is independent of the position of point P.
Sol.: Let the equation of the ellipse, with focus S being the pole, be

$$\frac{l}{r} = 1 + e\cos\theta. \tag{1}$$

Let α be the vectorial angle of P, then the vectorial angle of Q will be $\pi + \alpha$. Since P and Q lie on the conic (1), we have

$$\frac{l}{SP} = 1 + e\cos\alpha \text{ and } \frac{l}{SQ} = 1 + e\cos(\pi + \alpha) = 1 - e\cos\alpha.$$

This gives

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{1 + e\cos\alpha}{l} + \frac{1 - e\cos\alpha}{l} = \frac{2}{l} \quad \text{or} \quad \frac{SP}{SQ} = \frac{2}{l}\,SP - 1. \tag{2}$$

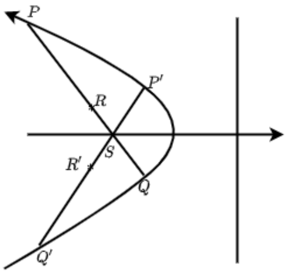


Figure 13:

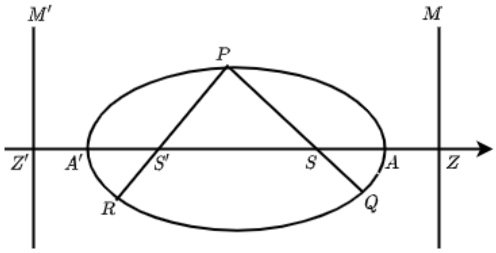


Figure 14:

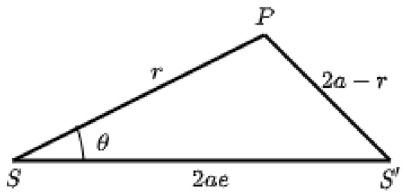


Figure 15:

Similarly, for the focal chord $PS'R$, we have

$$\frac{S'P}{S'R} = \frac{2}{l}\,S'P - 1. \tag{3}$$

Adding (2) and (3), we get

$$\frac{SP}{SQ} + \frac{S'P}{S'R} = \frac{2}{l}\,(SP + S'P) - 2.$$

But for an ellipse, $SP + S'P = 2a =$ length of major axis, therefore

$$\frac{PS}{SQ} + \frac{PS'}{S'R} = \frac{2}{l}\,(2a) - 2 = \frac{4a}{l} - 2 = \text{ a constant independent of } \alpha \text{ i.e. the position of } P.$$

5: A point P moves, so that the sum of its distance from two fixed points S and S' is constant and equal to $2a$. Show that P lies on the conic

$$\frac{a(1 - e^2)}{r} = 1 - e\cos\theta$$

referred to S as pole and SS' as initial line, SS' being equal to $2ae$.

Sol.: Taking S as pole and SS' as initial line, let the polar coordinates of point P be (r, θ). Then $SP = r$ and $\angle S'SP = \theta$.

Given that $SS' = 2ae$ and $SP + S'P = 2a$, from $\triangle SPS'$ in Figure (15), we have

$$\begin{aligned}
\cos\theta &= \frac{SP^2 + SS'^2 - S'P^2}{2SP \cdot SS'} \\
&= \frac{r^2 + (2ae)^2 - (2a - r)^2}{2r(2ae)} \\
&= \frac{r^2 + 4a^2e^2 - 4a^2 - r^2 + 4ar}{4rae} = \frac{a(e^2 - 1) + r}{re} \\
\therefore \quad e\cos\theta &= 1 + \frac{a(e^2 - 1)}{r} \quad \text{or} \quad \frac{a(1 - e^2)}{r} = 1 - e\cos\theta,
\end{aligned}$$

which is the required locus of P and is a conic.

6: A circle of diameter d passing through the focus of a conic, whose latus rectum is $2l$, meets the conic in four points whose distances from the focus are r_1, r_2, r_3, r_4, then prove that

$$\text{(i) } r_1 r_2 r_3 r_4 = \frac{d^2 l^2}{e^2}, \qquad \text{(ii) } \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{l}.$$

Sol.: Taking the focus as pole and the axis of conic as initial line, the equation of conic is

$$l/r = 1 + e\cos\theta. \tag{1}$$

The equation of the circle passing through the focus may be taken as

$$r = d\cos(\theta - \alpha) = d(\cos\theta\cos\alpha + \sin\theta\sin\alpha), \tag{2}$$

where d is the diameter of the circle and α is the angle which the diameter through the pole makes with the initial line. Eliminating θ from (1) and (2), we get

$$r = d\left[\frac{l-r}{er}\cos\alpha + \sqrt{1 - \left(\frac{l-r}{er}\right)^2}\sin\alpha\right]$$

$$\text{or} \quad \left[r - \frac{l-r}{er}d\cos\alpha\right]^2 = d^2\sin^2\alpha\left[1 - \left(\frac{l-r}{er}\right)^2\right]$$

$$\text{or} \quad e^2r^4 + (l-r)^2d^2\cos^2\alpha - 2er^2(l-r)d\cos\alpha = d^2\sin^2\alpha\left[e^2r^2 - (l-r)^2\right]$$

$$\text{or} \quad e^2r^4 + 2ed\cos\alpha r^3 + (d^2 - 2edl\cos\alpha - e^2d^2\sin^2\alpha) - 2ld^2r + d^2l^2 = 0,$$

which is an equation of degree four in r, and its roots give the focal distances r_1, r_2, r_3, r_4 of four points of intersection of conic and circle. Hence, by theory of equation, we have

$$r_1r_2r_3 + r_2r_3r_4 + r_1r_2r_4 + r_3r_4r_1 = -\frac{\text{coeeficient of } r}{\text{coeeficient of } r^4} = \frac{2ld^2}{e^2}, \tag{*}$$

$$\text{and} \quad r_1r_2r_3r_4 = \frac{\text{coeeficient of } r^0}{\text{coeeficient of } r^4} = \frac{d^2l^2}{e^2}, \tag{**}$$

The relation (**) gives part (i). Diving the relation (*) by (**), we get

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{l}.$$

Exercises

Q1. If PSP' and QSQ' are two perpendicular focal chord of a conic, then prove that $\frac{1}{PS.SP'} + \frac{1}{QS.SQ'}$ is constant.

Q2. A chord PQ of a conic whose eccentricity is e and semi-latus rectum l subtends a right angle at the focus S, show that

$$\left(\frac{1}{SP} - \frac{1}{l}\right)^2 + \left(\frac{1}{SQ} - \frac{1}{l}\right)^2 = \frac{e^2}{l^2}.$$

Q3. Prove that the perpendicular focal chords of a rectangular hyperbola are equal. (**Hint:** For a rectangular hyperbola, eccentricity $e = \sqrt{2}$.)

Q4. Determine the nature, latus-rectum and eccentricity of the following conics: (i) $\frac{15}{r} = 3 - 4\cos\theta$ (ii) $\frac{3}{r} = 2 + \sqrt{3}\cos\theta + \sin\theta$

Q5. Find the point on the conic $\frac{14}{r} = 3 - 8\cos\theta$, whose radius vector is 2.