Module-IV

Subject- Mathematics

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Topic: Two-dimensional Geometry

Subtopic: Polar Equation of Conics

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Module 4: Polar Equation of Pair of Tangents, Chord of contact, Polar and Normals of Conics

1 Pair of tangents

To find the equation of a pair of tangents to the conic $l/r = 1 + e\cos\theta$ drawn from an exterior point $P(r', \theta')$.

Let us suppose that α be the vectorial angle of the point of contact of any one of the tangents, say PQ, drawn from $P(r', \theta')$ to the conic $l/r = 1 + e\cos\theta$. The equation of the tangent at the point $'\alpha'$ of the conic is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha). \tag{1}$$

Since this tangent passes through (r', θ'), we have

$$\frac{l}{r'} = e\cos\theta' + \cos(\theta' - \alpha). \tag{2}$$

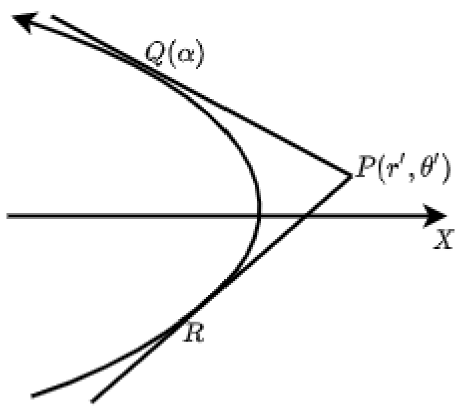

Figure 1: Pair of tangents

The equation of the pair of tangents will be obtained by eliminating α between (1) and (2). Using (1) and (2), we get

$$\left(\left(\frac{l}{r} - e\cos\theta\right)^2 - 1\right)\left(\left(\frac{l}{r'} - e\cos\theta'\right)^2 - 1\right) = \left(\cos^2(\theta - \alpha) - 1\right)\left(\cos^2(\theta' - \alpha) - 1\right)$$

$$\text{or}\quad \left(\left(\frac{l}{r} - e\cos\theta\right)^2 - 1\right)\left(\left(\frac{l}{r'} - e\cos\theta'\right)^2 - 1\right) = \sin^2(\theta - \alpha)\sin^2(\theta' - \alpha). \tag{3}$$

Again, from (1) and (2), we get

$$\begin{aligned}\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r'} - e\cos\theta'\right) - \cos(\theta - \theta') &= \cos(\theta - \alpha)\cos(\theta' - \alpha) - \cos(\theta - \theta')\\ &= \frac{1}{2}\left[\cos(\theta + \theta' - 2\alpha) + \cos(\theta - \theta')\right] - \cos(\theta - \theta')\\ &= \frac{1}{2}\left[\cos(\theta + \theta' - 2\alpha) - \cos(\theta - \theta')\right]\\ &= -\sin(\theta - \alpha)\sin(\theta' - \alpha)\end{aligned}$$

$$\text{or}\quad \left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{\rho} - e\cos\theta'\right) - \cos(\theta - \theta') = -\sin(\theta - \alpha)\sin(\theta' - \alpha) \tag{4}$$

Hence in view of (3) and (4), the required equation of the pair of tangents is given by

$$\boxed{\left(\left(\frac{l}{r} - e\cos\theta\right)^2 - 1\right)\left(\left(\frac{l}{r'} - e\cos\theta'\right)^2 - 1\right) = \left(\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r'} - e\cos\theta'\right) - \cos(\theta - \theta')\right)^2}.$$

Note: If $S = \left(\frac{l}{r} - e\cos\theta\right)$, $S' = \left(\frac{l}{r'} - e\cos\theta'\right)$, then the above equation is written as

$$(S^2 - 1)(S'^2 - 1) = [SS' - \cos(\theta - \theta')]^2.$$

2 Equation of Chord of contact

To find the equation of chord of contact of tangents from the point $P(\rho, \phi)$ ***to the conic*** $l/r = 1 + e\cos\theta$***.***

Let Q and R be the points of contact of the tangents drawn from the point $P(\rho, \phi)$ to the conic $l/r = 1 + e\cos\theta$. Then the chord QR is the chord of contact of the point P with respect to the given conic.

Let α and β be the vectorial angles of Q and R respectively. Then the tangents at $Q(\alpha)$ and $R(\beta)$ are given by

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha), \tag{1}$$

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \beta). \tag{2}$$

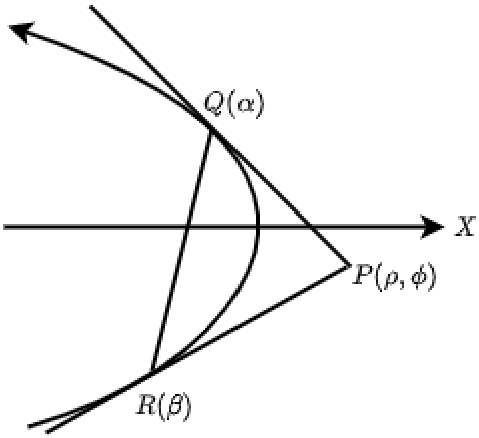

Figure 2: Chord of Contact

Since $P(\rho, \phi)$, the point of intersection of tangents, lies on both (1) and (2) , therefore we get

$$\frac{l}{\rho} = e\cos\phi + \cos(\phi - \alpha) \quad \text{and} \quad \frac{l}{\rho} = e\cos\phi + \cos(\phi - \beta) \tag{3}$$

which gives
$$e\cos\phi + \cos(\phi - \alpha) = e\cos\phi + \cos(\phi - \beta)$$

$$\text{or} \quad \cos(\phi - \alpha) = \cos(\phi - \beta) \quad \text{or} \quad (\phi - \alpha) = \pm(\phi - \beta).$$

Since $\alpha \neq \beta$, therefore we get $(\phi - \alpha) = -(\phi - \beta)$ or $\phi = \frac{\alpha+\beta}{2}$.
Putting this value of ϕ in one of the relations in (3), we get

$$\frac{l}{\rho} = e\cos\left(\frac{\alpha+\beta}{2}\right) + \cos\left(\frac{\beta-\alpha}{2}\right) \quad \text{or} \quad \cos\left(\frac{\beta-\alpha}{2}\right) = \left(\frac{l}{\rho} - e\cos\left(\frac{\alpha+\beta}{2}\right)\right). \tag{4}$$

Now, the equation of the chord QR is

$$\frac{l}{r} = \sec\left(\frac{\beta-\alpha}{2}\right)\cos\left(\theta - \frac{\alpha+\beta}{2}\right) + e\cos\theta,$$

$$\text{or} \quad \left(\frac{l}{r} - e\cos\theta\right)\cos\left(\frac{\beta-\alpha}{2}\right) = \cos\left(\theta - \frac{\alpha+\beta}{2}\right). \tag{5}$$

Using $\phi = \frac{\alpha+\beta}{2}$ and (4) in (5), we get the equation of the chord of contact of tangents from the point $P(\rho, \phi)$ as

$$\boxed{\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{\rho} - e\cos\phi\right) = \cos(\theta - \phi).}$$

3 Equation of Polar

To find the equation of the polar of a given point (ρ, ϕ) ***with respect to the conic*** $l/r = 1 = e\cos\theta$***.***

Polar: *The locus of the point of intersection of the tangents at the extremities of the chords on the conic which are drawn through a fixed point, is called the polar of that point.*

Let PQ be a chord drawn through a given fixed point $R(\rho, \phi)$ such that α and β are the vectorial angles of the extremities P and Q. Then the equation of the chord PQ is

$$\frac{l}{r} = \sec\left(\frac{\beta-\alpha}{2}\right)\cos\left(\theta - \frac{\alpha+\beta}{2}\right) + e\cos\theta.$$

Since this chord passes through the point $R(\rho, \phi)$, we have

$$\frac{l}{\rho} = \sec\left(\frac{\beta-\alpha}{2}\right)\cos\left(\phi - \frac{\alpha+\beta}{2}\right) + e\cos\phi,$$

$$\text{or} \quad \left(\frac{l}{\rho} - e\cos\phi\right)\cos\left(\frac{\beta-\alpha}{2}\right) = \cos\left(\phi - \frac{\alpha+\beta}{2}\right). \tag{1}$$

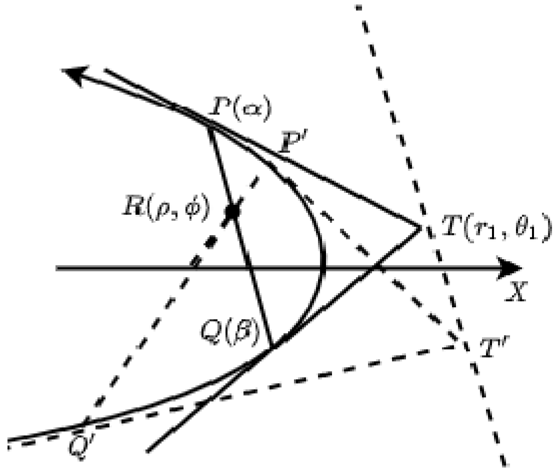

Figure 3: Polar TT'

If $T(r_1, \theta_1)$ is the point of intersection of the tangents at the points $P(\alpha)$ and $Q(\beta)$, then proceeding as in

$$\theta_1 = \frac{\alpha+\beta}{2} \quad \text{and} \quad \frac{l}{r_1} = e\cos\left(\frac{\alpha+\beta}{2}\right) + \cos\left(\frac{\beta-\alpha}{2}\right). \tag{2}$$

Using (2) in (1), we get

$$\left(\frac{l}{\rho} - e\cos\phi\right)\left(\frac{l}{r_1} - e\cos\theta_1\right) = \cos(\phi - \theta_1)$$

Thus, the locus of $T(r_1, \theta_1)$, that is, the equation of polar is

$$\boxed{\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{\rho} - e\cos\phi\right) = \cos(\theta - \phi)}$$

Remark: The polar of a point with respect to a given conic is the same as the chord of contact of the tangents drawn from that point to the conic, when the point lies outside the conic.

4 Equation of Normal

To find the equation of normal at a point having vectorial angle α on conic.

Let the equation of the conic be

$$\frac{l}{r} = 1 + e\cos\theta. \tag{1}$$

Let P be a point on conic having vectorial angle α, then the polar coordinates of P are $\left(\frac{l}{1+e\cos\alpha}, \alpha\right)$.

The equation of tangent PT at the point $'\alpha'$ on conic is

$$\frac{l}{r} = \cos(\theta - \alpha) + e\cos\theta. \tag{2}$$

Now as we know that a normal is a straight line which is perpendicular to the tangent at the point of contact,

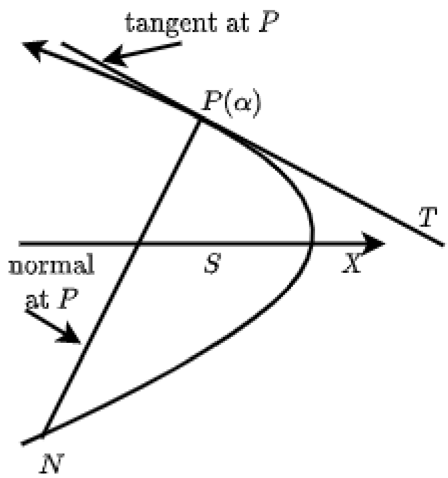

Figure 4: Normal PN

the equation of the normal PN at point $'\alpha'$ on conic may be obtained by replacing θ by $\frac{\pi}{2}+\theta$, and therefore, can be taken as

$$\frac{L}{r} = \cos\left(\frac{\pi}{2} + \theta - \alpha\right) + e\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta - \alpha) - e\sin\theta, \tag{3}$$

where L is to be determined so that the point P lies on it. Since the normal passes through P, we have

$$\frac{L}{\left(\frac{l}{1+e\cos\alpha}\right)} = -\sin(\alpha - \alpha) - e\sin\alpha \quad \text{or} \quad L = -\frac{el\sin\alpha}{1+e\cos\alpha}$$

Putting the value of L in (3), the equation of the normal at the point $'\alpha'$ is given by

$$\boxed{\left(\frac{e\sin\alpha}{1+e\cos\alpha}\right)\frac{l}{r} = \sin(\theta - \alpha) + e\sin\theta.}$$

Particular cases: If the conic is $l/r = 1 + e\cos(\theta - \gamma)$, then the equation of the normal at the point $P(\alpha)$ is

$$\left(\frac{e\sin(\alpha - \gamma)}{1+e\cos(\alpha - \gamma)}\right)\frac{l}{r} = \sin(\theta - \alpha) + e\sin(\theta - \gamma).$$

Solved Examples

Example 1. Normals at four points $\alpha, \beta, \gamma, \delta$ on the conic $l/r = 1 + e\cos\theta$ meet at a point (ρ, ϕ), then prove that

(a) $\tan\frac{\alpha}{2}\ \tan\frac{\beta}{2}\ \tan\frac{\gamma}{2}\ \tan\frac{\delta}{2} + \left(\frac{1+e}{1-e}\right)^2 = 0$, (b) $\alpha + \beta + \gamma + \delta - 2\phi = (2n+1)\pi$.

Solution.

Example 2. If the normal drawn at one end L of latus rectum of conic meets again the conic at point Q, then prove that

$$SQ = \frac{l(1+3e^2+e^4)}{1+e^2-e^4}.$$

Solution.

Example 3. If the normal to the conic $l/r = 1 + e\cos\theta$ at a point $'\alpha'$ meets the conic at point $'\beta'$ again, then prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = -\frac{1 + 2e\cos^2\frac{\alpha}{2} + e^2}{1 - 2e\sin^2\frac{\alpha}{2} + e^2}$$

Solution.

Example 4. If the normals at three points α, β, γ on the parabola $l/r = 1 + \cos(\theta - \psi)$ meet at a point (ρ, ϕ), then show that $2\phi = \alpha + \beta + \gamma - \psi$.

Solution.

Example 5. Three normals are drawn from a point to a parabola $l/r = 1 + \cos\theta$. Show that the distance of the point from the focus of the parabola is equal to the diameter of the circumcircle of the triangle formed by tangents at the three feet of the normals.

Solution.

Example 6. Find the locus of the pole of the chord which subtends the constant angle 2α at the focus of the conic $l/r = 1 + e\cos\theta$.

Solution.