Motion under Central force

Angular Momentum and Torque :-

The angular momentum of a particle is defined as the moment of its linear momentum.

Thus angular momentum $\vec{J}$ of a particle about a point is defined as -

$$\vec{J} = \vec{r} \times \vec{p} \qquad \text{——— (1)}$$

Where $\vec{r} \rightarrow$ vector distance of ~~point~~ particle from the point.

$\vec{p} = m\vec{v} \rightarrow$ momentum of the particle.

~~For circular motion,~~

Differentiating Eqn. (1) w.r.t. t we get -

$$\frac{d\vec{J}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times m\vec{v} + \vec{r} \times \vec{F} \qquad \left[\because \vec{F} = \frac{d\vec{p}}{dt}\right.$$

$$= 0 + \vec{r} \times \vec{F} \qquad \{\because \vec{v} \times \vec{v} = 0$$

The vector product of $\vec{r}$ and $\vec{F}$ is called moment of force or torque (बल आघूर्ण) ~~τ about~~ about the reference point, thus $\vec{\tau} = \frac{d\vec{J}}{dt} = \vec{r} \times \vec{F}$ ——— (2)

Thus torque is the rate of change of angular momentum $\vec{J}$.

Motion under central force and areal velocity:

Central force: A force which is always directed towards or away from a fixed point.

The central force is represented by -

$$\vec{F} = f(r)\hat{r} \quad \text{—— (1)}$$

where $f(r)$ is a scalar function of distance r and $\hat{r} = \frac{\vec{r}}{r}$

The torque acting on the particle,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$= \vec{r} \times f(r)\hat{r}$$

$$= f(r)\left[\vec{r} \times \frac{\vec{r}}{r}\right] \quad \because \vec{r} \times \vec{r} = 0$$

$$= 0$$

If $\vec{J}$ be the angular momentum of the particle

Then $\vec{\tau} = \frac{d\vec{J}}{dt} = 0$

$\Rightarrow$ or $\vec{J} = \text{constant}$. —— (2)

Thus angular momentum of particle moving under central force remains conserved.

Now from $\vec{J} = \vec{r} \times m\vec{v}$ —— (3)

3

Since under central force $\vec{J}$ is constant both in magnitude and direction; and is always perpendicular to plane containing $\vec{r}$ and is

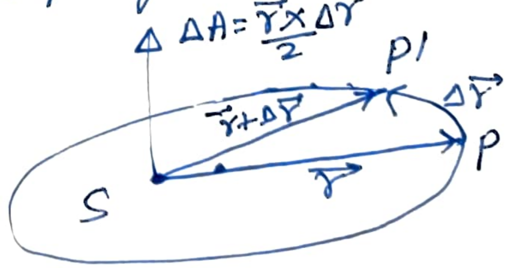
Thus path of a particle ~~under~~ moving under central force lie in a plane.

~~e.g. the motion of a pl~~

~~e.g.~~ The motion of a planet, say earth, around the Sun is an example of motion under central force as earth moves under the gravitational force (central force) $\vec{F}$ exerted by the sun.

Let S be the centre of the Sun and P be the planet ~~of~~ (mass m) in its orbit. Let $\vec{r}$ be the radius vector of the planet with respect to S. Suppose in small time interval Δt, planet moves from P to P' and radius vector becomes $\vec{r} + \Delta\vec{r}$.

Thus the area $\Delta\vec{A}$ swept by the radius vector in time interval Δt is



$$\Delta\vec{A} = \frac{1}{2}\vec{r} \times \Delta\vec{r}$$

Thus $$\frac{\Delta A}{\Delta t} = \frac{1}{2}\vec{r} \times \frac{\Delta\vec{r}}{\Delta t}$$

4

For $\Delta t \to 0$

$$\frac{d\vec{A}}{dt} = \frac{1}{2}\vec{r} \times \frac{d\vec{r}}{dt}$$

$$= \frac{1}{2}\vec{r} \times \vec{v}$$

$$= \frac{1}{2m}\vec{r} \times m\vec{v}$$

$$= \frac{\vec{J}}{2m}. \quad \text{——— (4)}$$

This expression gives areal velocity $\left(\frac{d\vec{A}}{dt}\right)$.

But $\vec{J}$ = constant for a central force.

$\therefore$ Areal velocity $= \frac{d\vec{A}}{dt} = \frac{\vec{J}}{2m}$ = constant.

i.e. the radius vector of the planet sweeps out equal area in equal time.

This is known as Kepler's second law

Kepler's Laws:

1. The Law of elliptical orbit: Each planet moves in an elliptical orbit around the sun. The sun being at one of the foci of the ellipse.
2. The Law of areas: The radius vector draw from the sun to a planet, sweeps out equal area in equal time i.e. the areal velocity of the radius vector is constant.

3. The harmonic law: The square of the period of revolution of any planet around the sun is proportional to the cube of the semi major axis of the ellipse.

Deduction of Kepler's laws from the Newton's Law of Gravitation:

Consider a planet of mass m is moving in the gravitational field of Sun. According to Newton's law of gravitation, the attractive force acting on the planet due to sun

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} \quad \text{——— (1)}$$

Where $M \rightarrow$ mass of the sun, $r \rightarrow$ distance of the planet from sun.

Since gravitational force is a central force, hence angular momentum $\vec{J}$ is conserved in magnitude and direction. Thus motion ~~must~~ of the planet must be in plane and the areal velocity $\frac{d\vec{A}}{dt} = \frac{\vec{J}}{2m}$ should be constant.

This is Kepler's 2nd Law.

The magnitude of areal velocity

$$\frac{dA}{dt} = \frac{J}{2m} \quad \text{——— (2)}$$

Also,

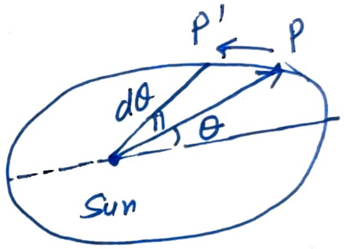
$$J = mvr$$
$$= mr^2\omega$$

or

$$r^2\omega = \frac{J}{m}, \text{ a constant} \quad \text{——— (3)}.$$

or

$$r^2\frac{d\theta}{dt} = \frac{J}{m} \quad \text{——— (4)}$$



The force on the planet is radial, thus from Newton's 2nd Law of motion -

$$\vec{F} = m \times \text{radial acceleration}$$
$$= m\left[\frac{d^2\vec{r}}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} \quad \text{——— (5)}$$

From (1) and (5)

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} \quad \text{——— (6)}$$

Since $\omega = \frac{d\theta}{dt}$ is also variable, but since the angular momentum J is constant, by using (4), $\omega = \frac{d\theta}{dt} = \frac{J}{mr^2}$

and (6) becomes,

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + \frac{J^2}{m^2r^3} \quad \text{——— (7)}$$

We only want to know the shape of the orbit, so all we need to do is find r in terms of θ.

7 ~~To~~ Also to obtain an equation which is easy to solve we make the substitution $r = \frac{1}{u}$. ——— (7a).

Now,

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

$$= -\frac{1}{u^2}\cdot\frac{du}{d\theta}\left(\frac{d\theta}{dt}\right)$$ [From (4), $\frac{d\theta}{dt} = \frac{J}{mr^2}$

$$= -\frac{1}{u^2}\cdot\frac{J}{mr^2}\cdot\frac{du}{d\theta}.$$ $\because r = \frac{1}{u}$.

$$= -\frac{J}{m}\frac{du}{d\theta}. \quad \text{———} \quad (8)$$

Differentiating again -

$$\frac{d^2r}{dt^2} = -\frac{J}{m}\cdot\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dt}$$

$$= -\frac{J}{m}\cdot\frac{d^2u}{d\theta^2}\cdot\frac{J}{mr^2}$$

$$= -\frac{J^2}{m^2}u^2\frac{d^2u}{d\theta^2} \quad \text{———} \quad (9)$$

Thus equation (7) becomes -

$$-\frac{J^2}{m^2}u^2\frac{d^2u}{d\theta^2} = -GMu^2 + \frac{J^2u^3}{m}$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{J^2}$$

$$\frac{d^2u}{d\theta^2} + \left(u - \frac{GMm^2}{J^2}\right) = 0$$

or

$$\frac{d^2}{d\theta^2}\left(u - \frac{GMm^2}{J^2}\right) + \left(u - \frac{GMm^2}{J^2}\right) = 0$$

$\because \frac{GMm^2}{J^2} = $ const.

$\frac{d^2}{d\theta^2}\left(\frac{GMm^2}{J^2}\right) = 0$

8 The solution is—

$$u - \frac{GMm^2}{J^2} = A\cos\theta$$

Where
$A \rightarrow$ Constant of integration

or $$\frac{J^2/GMm^2}{r} = \cancel{1+|A|\cos\theta}$$

$$\frac{J^2/GMm^2}{r} = 1 + \frac{J^2A}{GMm^2}\cos\theta \qquad \text{(10)}$$

This equation is of the form,

$$\frac{l}{r} = 1 + e\cos\theta \qquad \text{(11)}$$

Which represents a ~~col~~ conic section with eccentricity ~~A~~ $e = \frac{J^2A}{GMm^2}$ and semi latus rectum $l = \frac{J^2}{GMm^2}$.

Since the orbit of the ~~Sun~~ planet about the Sun must be closed, the total energy of the planet $E = \frac{1}{2}mv^2 - \frac{GMm}{r^2}$ should be negative,

E will be negative only if $e < 1$

Hence the orbit of the planet about the Sun is an ellipse, Which is Kepler's 1st law.

9

3rd Law

Consider semilatus rectum l of the elliptical orbit. If a and b be the semi-major and semi minor axis then -

$$l = \frac{b^2}{a} = \frac{J^2}{GMm^2} \quad \text{——— (12)}$$

If T be the time period of revolution of the planet about the Sun then -

$$T = \frac{\text{area of ellipse}}{\text{areal velocity}}$$

$$= \frac{\pi a b}{J/2m}$$

$$\therefore \quad T^2 = \frac{\pi^2 a^2 b^2 4m^2}{J^2}$$

$$= \frac{\pi^2 a^2 4\cancel{m^2}}{\cancel{J^2}} \cdot \frac{a \cancel{J^2}}{GM\cancel{m^2}} \quad \text{from (12)}$$

$$= \frac{4\pi^2}{GM} a^3$$

$$T^2 \propto a^3$$

The square of the period of revolution of the planet around the Sun is proportional to the cube of the semi-major axis of the ellipse.

Kepler's 3rd Law.