2016

Udai Pratap (Autonomous) College, Varanasi

E-learning Material

Module/ Lecture	**11**
Topic	**Fourier Equation of Heat flow**
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Rectilinear flow of heat along a bar : Fourier Equation of heat flow

Consider a long metal bar of uniform area of cross section 'A' heated at one end and heat flows along the length of bar. Suppose that the bas lies along x-axis whose origin lies at the hot end as shown in fig.1.

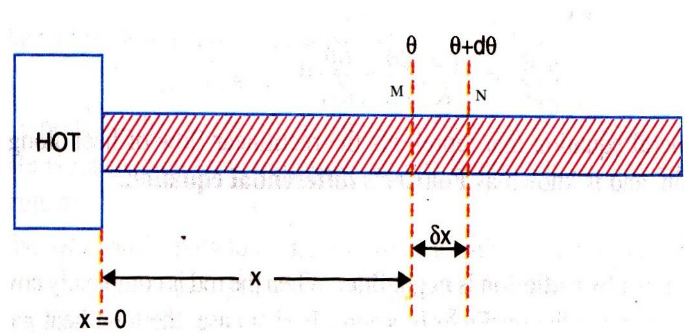

Fig. 1.

Let at a distance x from the hot end let θ be the temperature above the surrounding and $\frac{d\theta}{dx}$ be the temperature gradient, at any time. Consider two planes M and N perpendicular to the length of the bar at distance x and $x + \delta x$ from the hot end i.e. thickness of section MN will be δx . Also the temperature N will be $\theta + \frac{\partial \theta}{\partial x}\delta x$ and the temperature gradient at N will be $\frac{d}{dx}\left(\theta + \frac{\partial \theta}{\partial x}\delta x\right)$.

The heat flowing per second into the element at M is

$$Q_1 = -KA\frac{d\theta}{dx} \tag{1}$$

Where K is the coefficient of thermal conductivity.

Also, the heat flowing out per second from the element at N is

$$Q_2 = -KA\frac{d}{dx}\left(\theta + \frac{\partial \theta}{\partial x}\delta x\right)$$

$$= -KA\frac{d\theta}{dx} - KA\frac{\partial^2\theta}{\partial x^2}\delta x \quad (2)$$

Thus the heat gained by the section MN per second

$$Q = Q_1 - Q_2 = KA\frac{\partial^2\theta}{\partial x^2}\delta x \quad (3)$$

Before the steady state is reached, the quantity of Q is used (i) partially to raise the temperature of the bar say, Q_i and (ii) partially radiated out, say Q_r, i.e.

$$Q = Q_i + Q_r \quad (4)$$

Let $\frac{\partial\theta}{\partial t}$ be the rate of rise of temperature of rod at section MN. The heat used per second to raise the temperature of the bar at section MN will be

Q_i=mass X specific heat X rate of increase of temperature

$$Q_i = (A\delta x)\rho s\frac{\partial\theta}{\partial t}$$

$$Q_i = (A\delta x)\rho s\frac{\partial\theta}{\partial t} \quad (5)$$

where ‘A’ is the area of cross section of bar, ρ is the density of material of bar, s is the specific heat of the material of bar.

The heat lost due to radiation will be

$$Q_r = E\,p\,\delta x\,\theta \quad (6)$$

Where E is the emissive power of the surface of bar, p is the perimeter of the surface, θ be the average excess temperature above the surrounding of the bar between M & N.

Thus from Equations (3, 4, 5 & 6) we get,

$$KA\frac{\partial^2\theta}{\partial x^2}\delta x = (A\delta x)\rho s\frac{\partial\theta}{\partial t} + E\,p\,\delta x\,\theta$$

i.e. $$\frac{\partial^2\theta}{\partial x^2} = \frac{\rho s}{K}\frac{\partial\theta}{\partial t} + \frac{Ep}{KA}\theta \qquad (7)$$

This is the general equation for rectilinear flow of heat along a bar and it is known as Fourier Equation of Heat flow.

Now we will consider some particular cases of interest

1. **Steady State:** In this state, which is attain after some time, temperature of section MN does not change with time, i.e.

$$\frac{\partial\theta}{\partial t} = 0$$

Thus Equation (7) reduces to

$$\frac{\partial^2\theta}{\partial x^2} - \frac{Ep}{KA}\theta = 0 \qquad \textbf{(8)}$$

We see that θ is the function of x only and we consider following two cases:

(i). Radiation from the surface of the bar is allowed: If the bar is not insulated the Eq.(8) is

$$\frac{\partial^2\theta}{\partial x^2} - \frac{Ep}{KA}\theta = 0$$

$$\frac{\partial^2\theta}{\partial x^2} - \alpha^2\theta = 0 \qquad (9)$$

Where $\alpha^2 = \frac{Ep}{KA}$

General solution of the equation (9) is

$$\theta = b_1 e^{\alpha x} + b_2 e^{-\alpha x} \qquad (10)$$

Where b_1 and b_2 are constants which can be determined by knowing the boundary conditions. Let the boundary conditions

For x=0, $\theta = \theta_0$ and for x=∞ , $\theta = 0$ (i.e. the cold end is at the temperature of surrounding). Using these boundary conditions we get b_1=0 and $b_2 = \theta_0$ and from Eq. (10) we get

$$\theta = \theta_0 e^{-\alpha x} \tag{11}$$

Eq. (11) gives the excess of temperature above the surrounding at any point at distance x along the bar, exposed to the radiation, after steady state is reached. We see that θ is an exponential function of distance x of the point from the hot end and is independent of time.

(ii). Radiation from the surface of the bar is not allowed: If the bar is insulated, there will be no loss of heat from the surface of the rod, i.e. E=0 in Eq. (8), and from (8) we get

$$\frac{\partial^2 \theta}{\partial x^2} = 0 \ldots\ldots\ldots\ldots \tag{12}$$

On solving Eq. (9) we get

$$\theta = c_1 x + c_2 \tag{13}$$

Where c_1 and c_2 are constants which can be determined by knowing the temperatures at the two points on the bar,

If x=0, $\theta = \theta_0$ and at x=l, $\theta = \theta_1$ then $c_2 = \theta_0$.and $c_1 = -\frac{\theta_0 - \theta_1}{l}$

and therefore

$$\theta = \theta_0 - \frac{(\theta_0 - \theta_1)}{l} x \tag{14}$$

Equations (14) gives the excess temperature above surrounding at any point x along the insulated bar.

2. **Variable State:** If the heat lost by radiation is negligible(E=0), the equation **(7)** reduces to

$$\frac{\partial^2\theta}{\partial x^2} - \frac{\rho s}{K}\frac{\partial\theta}{\partial t} = 0$$

or

$$\frac{\partial\theta}{\partial t} = h\frac{\partial^2\theta}{\partial x^2} \qquad (15)$$

Quantity $h = \frac{K}{\rho s}$ is called diffusivity which determine s the rate at which temperature change takes place in the bar. The above equation can be solved when variation of θ with time is known.